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ARTICLE

A unified approach to the construction of higher-order derivative-free iterative methods for solving systems of nonlinear equations

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Abstract: : In this article, we introduce a unified approach to constructing a higher-order derivative-free scheme based on the approximations of $F'(z_k)^{-1}$. A family of order $p = 6,7$ derivative-free method is proposed and compared to some well-known methods. The necessary and sufficient condition for p-th order of convergence are given in terms of parameter matrices $\bar{\tau}^{(k)}$ and $\alpha^{(k)}$. Some good choices of $\bar{\tau}^{(k)}$ and $\alpha^{(k)}$ are offered. Numerical experiments were carried out to confirm the theoretical results.

Keywords: *Nonlinear systems; Higher order methods; Derivative-free methods; Order of convergence;*

INTRODUCTION

The issue of finding solution to $F(x) = 0$, where $F: D \subset R^m \to R^m$, D is open convex domain in R^m is an important and interesting task in both numerical analysis and applied scientific branch [1]-[21]. The most common used method for solving this problem is the second order Newton's method:

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1} F(x^{(k)}), \quad k = 0,1, \dots, \tag{1}$$

where x_0 is the initial guess and $F'(x)^{-1}$ is the inverse of Fréchet derivative $F'(x)$ of the function $F(x)$. In many practical situations, the derivative $F'(x)$ does not exist or is tedious to calculate, instead of the (1) often used derivative-free methods. For example, the quadratically convergent Traub-Steffensen method is given below [19], [20].

$$x^{(k+1)} = x^{(k)} - [F; w^{(k)}, x^{(k)}]^{-1} F(x^{(k)}), \quad k = 0,1, \dots, \tag{2}$$

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where $[F; w^{(k)}, x^{(k)}]$ is the first-order divided difference of F and $w^{(k)} = x^{(k)} + \gamma F(x^{(k)})$, γ is an arbitrary non-zero constant. $[F; w^{(k)}, x^{(k)}]^{-1}$ is the inverse of matrix $[F; w^{(k)}, x^{(k)}]$. The divided difference of F is a mapping $[\cdot,\cdot: F]: D \times D \subset R^m \times R^m \to L(R^m)$ defined by

$$[F; x + h_1, x] = \int_0^1 F'(x + th_1)dt, \ \forall x, h_1 \in R^m. \quad (3)$$

Expanding $F'(x + th_1)$ in Taylor series at the point x and integrating, we have

$$[F; x + h_1, x] = F'(x) + \frac{1}{2}F''(x)h_1 + \frac{1}{6}F'''(x)h_1^2 + O(h_1^3), \quad (4)$$

where $h_1^i = (h_1, h_1, \overset{i}{\ldots}, h_1)$. Over the years, many efficient derivative-free high-order methods have been proposed for solving nonlinear systems, see [1]-[16],[8]-[17] and references therein. The aim of this article is to develop a unified approach to constructing high-order derivative-free methods.

METHODS

Derivative-free iterative methods

We considered three-step symmetric iterations:

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F; w^{(k)}, s^{(k)}]^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - \overleftarrow{\tau}^{(k)}[F; w^{(k)}, s^{(k)}]^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \alpha^{(k)}[F; w^{(k)}, s^{(k)}]^{-1}F(z^{(k)}), \end{aligned} \quad (5)$$

where

$$\overleftarrow{\tau}^{(k)} = I + 2\Theta_k + O(h^2). \quad (6)$$

and

$$\alpha^{(k)} = I + 2\Theta_k + 4\Theta_k^2 - C_k + D_k + O(h^3), \quad (7)$$

here

$$C_k = \frac{1}{6}F'(x^{(k)})^{-1}F'''(x^{(k)})\left(3\left(F'(x^{(k)})^{-1}F(x^{(k)})\right)^2 - \gamma^2 F(x^{(k)})^2\right), \quad (8)$$

$$D_k = F'(x^{(k)})^{-1}F''(x^{(k)})\Theta_k F'(x^{(k)})^{-1}F(x^{(k)}). \quad (9)$$

Theorem 1. *Let $F: D \subset R^m \to R^m$ be a sufficiently Fréchet differentiable in an open convex set D containing simple solution α. Suppose that $F'(x)$ is continuous and nonsingular in $x = \alpha$ of $F(x) = 0$. Let, $x^{(0)}$ be an initial approximation, which is sufficiently close to α. Then the convergence order of iteration (5) is equal to seven, if parameter matrices $\overleftarrow{\tau}^{(k)}$ and $\alpha^{(k)}$ satisfy the conditions (6) and (7) respectively.*

We assume that $\gamma = 0$ and

$$D_k = 2\Theta_k^2 + O(h^3) \quad (10)$$

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Then the formula (7) leads to

$$\alpha_k = I + 2\Theta_k + 3d_k + 6\Theta_k^2 + O(h^3),$$

$$d_k = -\frac{1}{6}F'(x_k)^{-1}F'''(x_k)(F'(x_k)^{-1}F(x_k))^2.$$

Hence, Theorem 1 is a derivative-free version of Theorem 3.1 for $p = 7$, see Table 1 in [22] . According to (4) we have

$$\begin{aligned}[F; y^{(k)}, x^{(k)}] &= F'(x^{(k)}) - \tfrac{1}{2}F''(x^{(k)})[F; w^{(k)}, s^{(k)}]^{-1}F(x^{(k)}) \\ &+ \frac{F'''(x^{(k)})}{6}([F; w^{(k)}, s^{(k)}]^{-1}F(x^{(k)}))^2 + O(h^3).\end{aligned} \tag{11}$$

It is easy to show that

$$[F; w^{(k)}, s^{(k)}] = F'(x^{(k)})\left(I + \tfrac{1}{6}F'(x^{(k)})^{-1}F'''(x^{(k)})\gamma^2 F(x^{(k)})^2\right) + O(h^3). \tag{12}$$

From (12) we get

$$\begin{aligned}[F; w^{(k)}, s^{(k)}]^{-1} = \; &\left(I - \tfrac{1}{6}F'(x^{(k)})^{-1}F'''(x^{(k)})\gamma^2 F(x^{(k)})^2\right)F'(x^{(k)})^{-1} \\ &+ O(h^3).\end{aligned} \tag{13}$$

Substituting (13) into (11) we obtain

$$\begin{aligned}[F; y^{(k)}, x^{(k)}] &= F'(x^{(k)})(I - \Theta_k + \tfrac{1}{6}F'(x^{(k)})^{-1}F'''(x^{(k)}) \\ &\cdot \left(F'(x^{(k)})^{-1}F(x^{(k)})\right)^2) + O(h^3).\end{aligned} \tag{14}$$

Analogously, we obtain

$$\begin{aligned}[F; z^{(k)}, x^{(k)}] = \; &F'(x^{(k)}) - \frac{1}{2}F''(x^{(k)})[F; w^{(k)}, s^{(k)}]^{-1}\left(F(x^{(k)}) + F(y^{(k)})\right) \\ &+ \frac{F'''(x^{(k)})}{6}([F; w^{(k)}, s^{(k)}]^{-1}\left(F(x^{(k)}) + F(y^{(k)})\right))^2 + O(h^3),\end{aligned}$$

in which we have used (6). It is easy to show that

$$F'(x^{(k)})^{-1}F(y^{(k)}) = \Theta_k F'(x^{(k)})^{-1}F(x^{(k)}) + O(h^3). \tag{15}$$

Using (15) and (13) in last expression we have

$$\begin{aligned}[F; z^{(k)}, x^{(k)}] = \; &F'(x^{(k)})(I - \Theta_k - \tfrac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})\Theta_k F'(x^{(k)})^{-1} \\ &\cdot F(x^{(k)}) + \tfrac{1}{6}F'(x^{(k)})^{-1}F'''(x^{(k)})(F'(x^{(k)})^{-1} \\ &\cdot F(x^{(k)}))^2) + O(h^3).\end{aligned} \tag{16}$$

From (14) and (16) we get

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$$[F;z^{(k)},x^{(k)}]-[F;y^{(k)},x^{(k)}]= -\frac{1}{2}F''(x^{(k)})\Theta_k F'(x^{(k)})^{-1}F(x^{(k)}) +O(h^3). \tag{17}$$

Analogously, we obtain

$$[F;y^{(k)},z^{(k)}]=F'(z^{(k)})+\frac{1}{2}F''(x^{(k)})\Theta_k F'(x^{(k)})^{-1}F(x^{(k)})+O(h^3), \tag{18}$$

in which we have used (6), (15), (13). From (17) and (18) it follows that

$$F'(z^{(k)})=[F;y^{(k)},z^{(k)}]+[F;x^{(k)},z^{(k)}]-[F;y^{(k)},x^{(k)}]+O(h^3). \tag{19}$$

The following three-step derivative-free method was considered in [23]

$$\begin{aligned} y^{(k)} &= x^{(k)}-[F;w^{(k)},x^{(k)}]^{-1}F(x^{(k)}),\\ z^{(k)} &= y^{(k)}-\tilde{\tau}^{(k)}[F;w^{(k)},x^{(k)}]^{-1}F(y^{(k)})=\psi_4(x^{(k)},y^{(k)},w^{(k)}),\\ x^{(k+1)} &= z^{(k)}-\alpha^{(k)}[F;w^{(k)},x^{(k)}]^{-1}F(z^{(k)}). \end{aligned} \tag{20}$$

The same formula (19) was obtained for the iteration (20). As a consequence of (19) the following holds:

$$\alpha^{(k)}= ([F;y^{(k)},z^{(k)}]+[F;z^{(k)},x^{(k)}] -[F;y^{(k)},x^{(k)}])^{-1}[F;w^{(k)},s^{(k)}]+O(h^3), \tag{21}$$

$$\begin{aligned}\alpha^{(k)} &= F'(z^{(k)})^{-1}[F;w^{(k)},s^{(k)}]\\ &= [F;y^{(k)},z^{(k)}]^{-1}([F;w^{(k)},s^{(k)}] -[F;z^{(k)},x^{(k)}]+[F;y^{(k)},z^{(k)}])+O(h^3),\end{aligned} \tag{22}$$

and

$$\alpha^{(k)}= [F;y^{(k)},z^{(k)}]^{-1}([F;y^{(k)},z^{(k)}]-[F;z^{(k)},x^{(k)}] +[F;y^{(k)},x^{(k)}])[F;y^{(k)},z^{(k)}]^{-1}[F;w^{(k)},s^{(k)}]+O(h^3). \tag{23}$$

This means that formulae (63) and (65) in [23] hold true for iterations (5). The only difference is that $[F;w^{(k)},x^{(k)}]$ in (63) and (65) ([23]) is replaced by $[F;w^{(k)},s^{(k)}]$. Thus, we have the following theorem.

Theorem 2. *Assume that all the assumptions of Theorem 1 are satisfied. Then the iterations (5) have seventh-order convergence provided that $\tilde{\tau}^{(k)}$ satisfies the condition (6) and $\alpha^{(k)}$ is given by (21) or (22).*

Another way to obtain seventh-order iteration (5) is the direct application of the sufficient convergence condition (7) and $[F;y^{(k)},z^{(k)}]$ given by

$$[F;y^{(k)},z^{(k)}]= F'(y^{(k)})(I-\frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})\Theta_k \cdot F'(x^{(k)})^{-1}F(x^{(k)}))+O(h^3), \tag{24}$$

It is easy to show that

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$$\begin{aligned} F'(y^{(k)}) = \quad & F'(x^{(k)})\left(I - 2\Theta_k + \tfrac{1}{2}F'(x^{(k)})^{-1}F'''(x^{(k)})\right) \\ & \cdot\left(F'(x^{(k)})^{-1}F(x^{(k)})\right)^2 + O(h^3). \end{aligned} \tag{25}$$

for which we have used (4), (6), (13). Substituting (25) into (24) and using (13) we get

$$\widetilde{A_k} = [F; w^{(k)}, s^{(k)}]^{-1}[F; z^{(k)}, y^{(k)}] = I - 2\Theta_k + C_k - \tfrac{1}{2}D_k + O(h^3), \tag{26}$$

where C_k and D_k are determined by (8), (9). From (26) it follows that

$$\begin{aligned} \widetilde{A_k}^{-1} = \quad & [F; z^{(k)}, y^{(k)}]^{-1}[F; w^{(k)}, s^{(k)}] = I + 2\Theta_k - C_k \\ & + \tfrac{1}{2}D_k + 4\Theta_k^2 + O(h^3). \end{aligned} \tag{27}$$

From (7) and (27) it follows that

$$\widetilde{A_k}^{-1} = \alpha^{(k)} - \frac{1}{2}D_k + O(h^3).$$

Using formula (17) in last expression we have

$$\widetilde{A_k}^{-1} = \alpha^{(k)} + F'(x^{(k)})^{-1}\left([F; z^{(k)}, x^{(k)}] - [F; y^{(k)}, x^{(k)}]\right) + O(h^3).$$

We can replace $F'(x^{(k)})^{-1}$ by $[F; w^{(k)}, s^{(k)}]^{-1}$ due to (17) and hence from last expression we find

$$\begin{aligned} \alpha^{(k)} = \quad & [F; z^{(k)}, y^{(k)}]^{-1}[F; w^{(k)}, s^{(k)}] - [F; w^{(k)}, s^{(k)}]^{-1} \\ & \cdot\left([F; z^{(k)}, x^{(k)}] - [F; y^{(k)}, x^{(k)}]\right) + O(h^3). \end{aligned} \tag{28}$$

In [23] consider another option

$$\alpha^{(k)} = \left([F; y^{(k)}, z^{(k)}] + [F; z^{(k)}, x^{(k)}] - [F; y^{(k)}, x^{(k)}]\right)^{-1}[F; w^{(k)}, x^{(k)}]. \tag{29}$$

Since $\widetilde{A_k}^2 = I - 4\Theta_k + 4\Theta_k^2 + 2C_k - D_k + O(h^3)$ then $\widetilde{A_k}^{-1}$ can be seeking for

$$\widetilde{A_k}^{-1} = pI + q\widetilde{A_k} + s\widetilde{A_k}^2 + O(h^3). \tag{30}$$

Substituting $\widetilde{A_k}^{-1}$, $\widetilde{A_k}$ and $\widetilde{A_k}^2$ into last equality we have

$$\begin{aligned} I + 2\Theta_k - C_k + \frac{1}{2}D_k + 4\Theta_k^2 + O(h^3) = \quad & (p+q+s)I - 2(q+2s)\Theta_k \\ & + 4s\Theta_k^2 + (q+2s)C_k \\ & - \frac{1}{2}(q+2s)D_k + O(h^3). \end{aligned}$$

From this we obtain

$$q + 2s = -1,\ s = 1,\ p + q + s = 1 \Rightarrow p = 3,\ q = -3,\ s = 1.$$

Thus, substituting $\widetilde{A_k}^{-1} = 3I - 3\widetilde{A_k} + \widetilde{A_k}^2$ into (28) we obtain

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$$\alpha^{(k)} = 3I - 3\widetilde{A_k} + \widetilde{A_k}^2 - \left[F; w^{(k)}, s^{(k)}\right]^{-1}\left(\left[F; z^{(k)}, x^{(k)}\right] - \left[F; y^{(k)}, x^{(k)}\right]\right) + O(h^3). \quad (31)$$

We consider another $\bar{\tau}^{(k)}$

$$\begin{aligned}\bar{\tau}^{(k)} = \; & (2+a)I - \left[F; w^{(k)}, s^{(k)}\right]^{-1}\big((2a+b)\left[F; y^{(k)}, x^{(k)}\right] \\ & + b\left[F; y^{(k)}, w^{(k)}\right] - b\left[F; w^{(k)}, x^{(k)}\right] \\ & + (1-a-b)\left[F; y^{(k)} + cF\left(y^{(k)}\right), y^{(k)}\right]\big) + O(h^2),\end{aligned} \quad (32)$$

(a, b some constants) satisfies the condition (6). The obtained result can be formulated as follows:

Theorem 3. *Let us assume that all the assumptions of Theorem 1 are satisfied, then iteration (5) has a seventh-order of convergence when $\bar{\tau}^{(k)}$ satisfies the condition (32) and $\alpha^{(k)}$ is given by (31).*

To determine $\alpha^{(k)}$ given by (31) only one inverse matrix is needed, whereas in (28) two inverse matrices are required. Similar formula to (31) was obtained by Wang et al.[17] . Assume that (10) holds. Then (7) and (26) leads to

$$\alpha^{(k)} = I + 2\Theta_k + 6\Theta_k^2 - C_k + O(h^3). \quad (33)$$

and

$$\widetilde{A_k} = I - 2\Theta_k - \Theta_k^2 + C_k + O(h^3), \quad (34)$$

respectively, In this case $\alpha^{(k)}$ given by (33) can be expressed as

$$\alpha^{(k)} = p_1 I + q_1 \widetilde{A_k} + s_1 \widetilde{A_k}^2 + O(h^3). \quad (35)$$

Substituting (34) into (35) we obtain

$$\begin{aligned}I + 2\Theta_k + 6\Theta_k^2 - C_k = \; & (p_1 + q_1 + s_1)I - 2\Theta_k(q_1 + 2s_1) \\ & + (-q_1 + 2s_1)\Theta_k^2 + (q_1 + 2s_1)C_k + O(h^3),\end{aligned}$$

which holds when

$$p_1 + q_1 + s_1 = 1, q_1 + 2s_1 = -1, \; -q_1 + 2s_1 = 6 \Rightarrow s_1 = \frac{5}{4}, \; q_1 = -\frac{7}{2}, \; p_1 = \frac{13}{4}.$$

Thus, we get

$$\alpha^{(k)} = \tfrac{13}{4} I - \tfrac{7}{2}\widetilde{A_k} + \tfrac{5}{4}\widetilde{A_k}^2 + O(h^3). \quad (36)$$

The obtained result can be formulated as follows:

Theorem 4. *Let us assume that all the assumptions of Theorem 1 are satisfied, then iteration (5) has a seventh-order of convergence when $\bar{\tau}^{(k)}$ satisfies the condition (32) and $\alpha^{(k)}$ is given by (36).*

In [23], we consider other choices

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$$\bar{\tau}^{(k)} = 3I - 2\left[F; w^{(k)}, s^{(k)}\right]^{-1}\left[F; y^{(k)}, x^{(k)}\right]. \tag{37}$$

$$\bar{\tau}^{(k)} = 2I - \left[F; w^{(k)}, s^{(k)}\right]^{-1}\left[F; y^{(k)} + cF\left(y^{(k)}\right), y^{(k)}\right], \text{when } a = b = 0, \tag{38}$$

$$\begin{aligned}\bar{\tau}^{(k)} &= \left[F; w^{(k)}, x^{(k)}\right]^{-1}\left(\left[F; w^{(k)}, x^{(k)}\right] - D^{(k)}\right) + O(h^2)\\ &= 3I - \left[F; w^{(k)}, x^{(k)}\right]^{-1}\left(\left[F; w^{(k)}, y^{(k)}\right] + \left[F; y^{(k)}, x^{(k)}\right]\right).\end{aligned} \tag{39}$$

Note that the easy choices of (32) are (37) and (38). Whang et al. in [17] were obtained seventh-order derivative free iteration

$$\begin{aligned} y^{(k)} &= x^{(k)} - B^{-1}F\left(x^{(k)}\right),\\ z^{(k)} &= y^{(k)} - \left(3I - 2\left[F; w^{(k)}, s^{(k)}\right]^{-1}\left[F; y^{(k)}, x^{(k)}\right]\right)B^{-1}F\left(y^{(k)}\right),\\ x^{(k+1)} &= z^{(k)} - \alpha^{(k)}B^{-1}F\left(z^{(k)}\right),\end{aligned} \tag{40}$$

where $B = \left[F; w^{(k)}, s^{(k)}\right]$ and $\alpha^{(k)}$ given by (36). Thus, our iteration (5) with choices $\bar{\tau}^{(k)}$ and $\alpha^{(k)}$ given by (32) and (36) respectively includes the iteration (40) as special cases.

Now we consider another first order divided difference

$$Q = \left[F; u^{(k)}, v^{(k)}\right],\ u^{(k)} = z^{(k)} + bF\left(z^{(k)}\right),\ v^{(k)} = z^{(k)} - bF\left(z^{(k)}\right). \tag{41}$$

It is easy to show that

$$Q = F'\left(z^{(k)}\right) + O(h^3), \tag{42}$$

and

$$B^{-1}Q = B^{-1}F'\left(z^{(k)}\right) = I - 2\Theta_k + C_k - D_k + O(h^3). \tag{43}$$

Similarly, $\alpha^{(k)}$ can be expressed by linear combination of the form

$$\alpha^{(k)} = p_2 I + q_2 B^{-1}Q + r_2(B^{-1}Q)^2 + s_2(B^{-1}Q)^3 + O(h^3). \tag{44}$$

Substituting (7) and (43) into (44) we obtain

$$\begin{aligned} I + 2\Theta_k + 4\Theta_k^2 - C_k + D_k = \ & p_2 I + q_2(I - 2\Theta_k + C_k - D_k)\\ & + r_2\left(I - 4\Theta_k + 4\Theta_k^2 + 2(C_k - D_k)\right)\\ & + s_2\left(I - 6\Theta_k + 12\Theta_k^2 + 3(C_k - D_k)\right) + O(h^3),\end{aligned}$$

which holds when

$$\begin{aligned} p_2 + q_2 + r_2 + s_2 = 1, q_2 + 2r_2 + 3s_2 = -1,\ r_2 + 3s_2 = 1 \Rightarrow\\ q_2 = -3 + 3s_2,\ r_2 = 1 - 3s_2,\ p_2 = 3 - s_2.\end{aligned}$$

Hence

$$\begin{aligned}\alpha^{(k)} = \ & (3 - s_2)I - 3(1 - s_2)B^{-1}Q + (1\\ & -3s_2)(B^{-1}Q)^2 + s_2(B^{-1}Q)^3 + O(h^3).\end{aligned} \tag{45}$$

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Thus, we obtain the family of seventh-order derivative-free iteration (5) with $\alpha^{(k)}$ given by (45). The result can be formulated as:

Theorem 5. *Let us assume that all the assumptions of Theorem 1 are satisfied, then iteration (5) has a seventh-order of convergence when parameter matrix $\bar{\tau}^{(k)}$ satisfies the conditions (6) and $\alpha^{(k)}$ is given by (45).*

Thus, we obtain the family of seventh-order derivative-free iterations (5) with $\alpha^{(k)}$ given by (45). In particular, when $s_2 = 0$ the formula (45) leads to

$$\alpha^{(k)} = 3I - 3B^{-1}Q + (B^{-1}Q)^2. \quad (46)$$

When $s_2 = -\frac{5}{4}$ the formula (45) leads to

$$\alpha^{(k)} = \frac{17}{4}I - \frac{27}{4}B^{-1}Q + \frac{19}{4}(B^{-1}Q)^2 - \frac{5}{4}(B^{-1}Q)^3. \quad (47)$$

In this case the third-step of iteration (5) coincides with third-step of iteration proposed by Narang et al. in [6] . The scheme proposed by Narang et al. in [6] differs from (5) only by a second step with $\bar{\tau}^{(k)} = I$.

Remark 1. *For seventh-order iterations (5) one can choose $\bar{\tau}^{(k)}$ by formula (32).*

Similar results to Theorem 4 and Theorem 5 can be obtained for iteration (20) using $[F; w^{(k)}, x^{(k)}]$ instead of $[F; w^{(k)}, s^{(k)}]$. For example, Theorem 5 can be formulated for iteration (20) as follows:

Theorem 6. *Let us assume that all the assumptions of Theorem 1 are satisfied, then iteration (20) has a seventh-order of convergence when $\bar{\tau}^{(k)}$ satisfies the condition*

$$\bar{\tau}^{(k)} = I + 2\Theta_k + A_k + O(h^2) \text{ and } \alpha^{(k)} \text{ is given by (46), where}$$

$$B^{-1}Q = [F; w^{(k)}, x^{(k)}]^{-1}[F; v^{(k)}, z^{(k)}],\ v^{(k)} = z^{(k)} + bF(z^{(k)}). = z^{(k)} + bF(z^{(k)}).$$

The proof of Theorem 6 is the same as proof of Theorem 4 and hence, it is omitted here. Such theorem was proven by Sharma et al in [14] by means of symbolic computation. The selected two fourth order iterations in [14] are obtained from $\bar{\tau}^{(k)} = aI + G^{(k)}\left((3-2a)I + (a-2)G^{(k)}\right) + O(h^2)$ [23] with $a = 2$ and $a = 0$ respectively. Thus, we obtained many iterative methods with convergence order six and seven.

Computational Cost of methods

We now discuss the computational cost of the considered methods. In Table 1, we applied NLS1 to denote the number of linear system with some coefficient matrices $[F; w^{(k)}, x^{(k)}]$ and $[F; w^{(k)}, s^{(k)}]$ for methods (20) and (5) respectively and by NLS2, we denoted the number of linear system with other coefficient matrices. NFE is the number of function evaluations, $M \times V$ is the matrix by vector multiplications, C is total computation cost per iterations. From those tables we see that the most effective and cheapest methods of order seven are S7, NM7 and (5) for which the total cost is $n^3/3$.

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Table 1 Values of C for seventh order methods

method	$\bar{\tau}^{(k)}$, α_k	NFE	NLS1	NLS2	$M \times V$	C
S7 [17]		$3n^2$	6	-	3	$\frac{1}{3}n^3+14n^2-\frac{1}{3}n$
NM7 [6]		$2n^2+3n$	5	-	3	$\frac{1}{3}n^3+11n^2+\frac{2}{3}n$
Whang et al [16]		$5n^2-n$	4	1	1	$\frac{2}{3}n^3+16n^2-\frac{5}{3}n$
Sharma et al [12]		$5n^2+n$	4	1	2	$\frac{2}{3}n^3+17n^2+\frac{1}{3}n$
(20)	(39), (29)	$5n^2+n$	4	1	1	$\frac{2}{3}n^3+16n^2+\frac{1}{3}n$
(20)	(39), (46)	$5n^2+n$	4	1	1	$\frac{2}{3}n^3+16n^2+\frac{1}{3}n$
	(37),(21)	$4n^2+n$	4	1	1	$\frac{2}{3}n^3+14n^2+\frac{1}{3}n$
	(37), (22)	$4n^2+n$	4	1	4	$\frac{2}{3}n^3+15n^2+\frac{1}{3}n$
(5)	(37), (31)	$4n^2+n$	4	1	4	$\frac{1}{3}n^3+18n^2+\frac{2}{3}n$
	(38), (31)	$5n^2$	6	-	3	$\frac{1}{3}n^3+16n^2-\frac{1}{3}n$
	(37), (46)	$2n^2+5n$	6	-	3	$\frac{1}{3}n^3+13n^2+\frac{17}{3}n$
	(37), (36)	$3n^2+2n$	6	-	3	$\frac{1}{3}n^3+15n^2+\frac{5}{3}n$

RESULTS

Numerical Results

The purpose of this section is to demonstrate the convergence behaviour of the proposed schemes (5) and (20). Additionally, we compared it with several well-known methods of the same order. To do this, consider a standard testing problem.

Example 1. *Consider the system of 20 nonlinear equations (see [6],[17]]):*

$$x_i - \cos\left(2x_i - \sum_{j=1}^{20} x_j\right) = 0, \quad 1 \le i \le 20.$$

The initial value assumed is $x_0 = \{-0.9, -0.9 \dots, -0.9\}^T$ *for obtaining the solution* $\alpha \approx \{-0.89, -0.89 \dots, -0.89\}^T$.

Example 2. *The second nonlinear system is given by [13],[17] .*

$$\begin{cases} x_i^2 x_{i+1} - 1 = 0, & i = 1, \dots, 99, \\ x_1 x_{100}^2 - 1 = 0. \end{cases}$$

The initial value is $x_0 = \{1.5, 1.5, \dots, 1.5\}^T$ *for the solution* $\alpha = \{1, 1, \dots, 1\}^T$.

The methods are tested by using the value -0.01 for parameter γ. Tables 2, 3 contain the following information: the required number of iterations (k), the errors in the last step $\| x^{(k)} - x^{(k-1)} \|$, the computational order of convergence ρ_k and the computational time $CPUtime$ (in seconds). The stopping criterion used is $\| x^{(k-1)} - x^{(k)} \| \le 10^{-150}$. To verify the theoretical order, we computed ρ_k using the formula [4].

$$\rho_k = \frac{\ln(\| x^{(k+1)} - x^{(k)} \| / \| x^{(k)} - x^{(k-1)} \|)}{\ln(\| x^{(k)} - x^{(k-1)} \| / \| x^{(k-1)} - x^{(k-2)} \|)}.$$

The numerical results displayed in Tables 2 and 3 are consistent with the theoretical results obtained in the preceding sections. In view of our analysis of the results in Tables 2, 3 given below, overall the best method is (5). Furthermore, the best choice of parameters $\bar{\tau}^{(k)}$ and $\alpha^{(k)}$ are (37) and (36).

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Table 2. Comparison of methods for example 1

Methods	$\bar{\tau}^{(k)}$	$\alpha^{(k)}$	k	$\| x_k - x_{k-1} \|$	ρ_k	CPUtime
(20)	(39)	(29)	5	0.1687e-783	6.99	59.05
(20)	(39)	(46)	5	0.3992e-747	6.99	67.08
(5)	(37)	(31)	5	0.4668e-676	7	50.87
(5)	(38)	(31)	5	0.9677e-951	6.99	49.75
(5)	(37)	(45)	5	0.5270e-707	6.99	79.91
(5)	(38)	(45)	5	0.2723e-999	7	79.85
(5)	(37)	(21)	5	0.3234e-700	6.99	52.17
(5)	(38)	(21)	5	0.1103e-873	7	40.23
(5)	(37)	(36)	5	0.1803e-610	6.99	40.24
(5)	(38)	(36)	5	0.6270e-719	7	52.02
Narang et.al [6]	-	-	5	0.1043e-886	6.99	53.37
Sharma et al [12]	-	-	5	0.3098e-797	6.99	78.88

Table 3. Comparison of methods for example 2

Methods	$\bar{\tau}^{(k)}$	$\alpha^{(k)}$	k	$\| x_k - x_{k-1} \|$	ρ_k	CPUtime
(20)	(39)	(29)	5	1.78E-182	6.99	159.73
(20)	(39)	(46)	5	0.3122e-751	6.99	177.88
(5)	(37)	(31)	5	0.5140e-621	7	160.44
(5)	(38)	(31)	4	0.8557e-958	6.99	140.78
(5)	(37)	(45)	5	0.6271e-417	6.99	199.97
(5)	(38)	(45)	5	0.1793e-560	7	189.12
(5)	(37)	(21)	5	0.4132e-702	6.99	164.56
(5)	(38)	(21)	5	0.1545e-879	7	160.98
(5)	(37)	(36)	5	0.2803e-710	6.99	150.27
(5)	(38)	(36)	5	0.1270e-817	7	162.23
Narang et.al [6]	-	-	5	0.1043e-886	6.99	163.37
Sharma et al [12]	-	-	5	0.3098e-797	6.99	188.11

CONCLUSIONS

We constructed several families of order $p = 6,7$ derivative free methods. The necessary and sufficient conditions for p-th order of convergence are defined in terms of parameter matrices. Based on the approximations for $F'(z_k)^{-1}$ we suggest some choices of these matrices. Numerical experiments are given to illustrate the theoretical results.

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