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Some modifications and extensions of Popovski's and Laguerre's families for solving systems of nonlinear equations

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Abstract. Two modifications of Popovski's and Laguerre's families of methods are developed, which free of second derivatives. The improved modifications have fourth-order of convergence. Moreover, we propose the extensions of modifications to solve nonlinear systems. The convergence order of two and three-step iterations equal to four, six and seven respectively. The numerical results confirm the order of convergence. In addition, we investigate the basin of attraction of the methods and its dependence on the convergence behavior. The comparison is made based on the performance on examples of nonlinear problems and the CPU time.

1. Introduction

Finding solutions to nonlinear equations $f(x) = 0$ and nonlinear systems $F(x) = 0$ is a fundamental problem in science and engineering. Recently, numerous iterative methods have been developed for solving nonlinear equations and systems [3, 4, 8, 17, 17, 21] and in the references therein. Many researchers have been interested in Popovski's and Laguerre's families to find simple and multiple roots of nonlinear equations because they include many well-known methods such as Chebyshev's method, Halley's method, Euler-Cauchy's method and Hansen-Patrick's method and so on [3, 4, 8, 10, 11, 13–15, 20]. Moreover, these families have some intersections, namely the Halley's and Euler-Cauchy's methods are included in both families. The main drawback of these families is that they include term with second derivative of the function. Note that in [9] proposed two modifications of Popovski's method free from second derivatives, but the convergence order equal to 3 and 2.732. It was shown that [3, 4, 8] all members of Popovski's families of methods and Euler-Cauchy, Halley, and Ostrowski's methods in Laguerre's family have no extraneous fixed points.

The main purpose of this paper is as follows:

– to suggest modifications of these families, free from second derivative
– to obtain an improved convergence order of methods
– to extend modifications to the multidimensional case.

The rest parts of this paper are structured as follows. In Sec. 1.1 we suggest modifications of methods. In Sec. 1.2 we consider improved modifications of Popovski's and Laguerre's families with fourth order of convergence. Extensions of these families with fourth and sixth order of convergence to solve systems of nonlinear equations are presented in Sec.

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2. Sec. 3 is devoted to the numerical experiments and to the analysis of the dynamical behavior, which exhibit the accuracy, efficiency and stability of modifications. Finally, the conclusion is drawn in the last section.

1.1. Modified Popovski's family of methods

Now, we consider iterative methods for finding a simple root x^* of $f(x) = 0$, where $f : I \subset \mathbb{R} \to \mathbb{R}$ is defined on an open interval I and $f'(x^*) \neq 0$. The third order Popovski's family of methods can be rewritten as [3, 8]:

$$x_{k+1} = x_k - \tau_k \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2 \dots \tag{1.1}$$

where

$$\tau_k = \frac{(1-r)}{2w_k}\left\{\left[1 - \frac{2r}{1-r}w_k\right]^{1/r} - 1\right\}, \quad w_k = \frac{f''(x_k)f(x_k)}{2f'(x_k)^2}, \tag{1.2}$$

where the parameter $r \neq 1$ and $r \neq 0$. Let

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)} \quad \text{and} \quad \theta_k = \frac{f(y_k)}{f(x_k)}. \tag{1.3}$$

Then it is easy to show that

$$\theta_k = \frac{f''(x_k)f(x_k)}{2f'(x_k)^2} + O(f(x_k)^2) = w_k + O(f(x_k)^2). \tag{1.4}$$

Using this approximation (1.4) in (1.1) we get

$$x_{k+1} = x_k - \tau_k \frac{f(x_k)}{f'(x_k)}, \tag{1.5}$$

where

$$\tau_k = \frac{(1-r)}{2\theta_k}\left\{\left[1 - \frac{2r}{1-r}\theta_k\right]^{1/r} - 1\right\}, \quad r \neq 1. \tag{1.6}$$

Further, the iteration (1.5) with parameter (1.6) will be called the modified Popovski's family of methods. Using the well known expansion

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \tag{1.7}$$

in (1.6) we get

$$\tau_k = 1 + \theta_k + \frac{2(1-2r)}{3(1-r)}\theta_k^2 + O(f(x_k)^3). \tag{1.8}$$

In [23] was proven that the iteration (1.5) has a fourth-order of convergence if and only if τ_k satisfies (see also [19])

$$\tau_k = 1 + \theta_k + 2\theta_k^2 + O(f(x_k)^3). \tag{1.9}$$

The comparison of (1.8) with (1.9) show that the convergence order ρ of modified Popovski's family of methods equal to

$$\rho = \begin{cases} 4 & \text{when } r = 2, \\ 3 & \text{when } r \neq 2. \end{cases} \tag{1.10}$$

Analogously, the modified generalization of Laguerre's family of method [20] for simple root reads as (1.5) with τ_k given by

$$\tau_k = W(\theta_k, \alpha, m) = \frac{\alpha+1}{\alpha + (1-m(\alpha+1)\theta_k)^{1/m}}, \tag{1.11}$$

where the parameters $\alpha \neq -1$ and $m \neq 0$. It is easy to show that the modified Popovski's method and modified generalization of Laguerre's method coincides when $r = m = 2$ and $\alpha = 1$.

In this case (1.6) and (1.11) coincides to each other i.e,

$$\tau_k = \frac{2}{1+\sqrt{1-4\theta_k}} = 1 + \theta_k + 2\theta_k^2 \cdots . \tag{1.12}$$

It means that the convergence order ρ of the modified generalization of Laguerre's family of methods equal to

$$\rho = \begin{cases} 4 & \text{when } \alpha = 1,\ m = 2 \\ 3 & \text{otherwise } (\alpha \neq 1,\ m \neq 2). \end{cases} \tag{1.13}$$

The modified Popovski's family of method (1.5), (1.6) for $r = 1/2$, $r = -1$ and $r = 2$ includes the modified Chebyshev's, Halley's and Euler-Cauchy's methods respectively. According to (1.10) the modified Euler-Cauchy's method has a fourth order of convergence. Analogously, modified Laguerre's method with $m = 2$, $\alpha = 1$ has a fourth order of convergence because of (1.13).

For (1.11) we have

$$\tau_k = W(\theta_k, \alpha, m) = 1 + \theta_k + \left(1 - \frac{1-m}{2}(\alpha+1)\right)\theta_k^2 + \cdots \tag{1.14}$$

From this and (1.9) we conclude that the convergence order of (1.5) with (1.11) equal to

$$\rho = \begin{cases} 4 & \text{when } (m-1)(\alpha+1) = 2 \\ 3 & \text{when } (m-1)(\alpha+1)) \neq 2. \end{cases}$$

According to [23], the iteration (1.5) has a third order of convergence if

$$\tau_k = \frac{1+c\theta_k}{1-(1-c)\theta_k}, \quad c \in \mathbb{R}. \tag{1.15}$$

1.2. Improved modifications of Popovski's and Laguerre's families with fourth order of convergence

We prove the following: The iteration (1.5) with parameter τ_k given by

$$\tau_k = \frac{(1-r)}{2\theta_k}\left\{\left[1 - \frac{2r}{1-r}\theta_k\right]^{1/r} - 1\right\} + \frac{2(2-r)}{3(1-r)}\theta_k^2, \quad r \neq 1. \tag{1.16}$$

and

$$\tau_k = W(\theta_k, \alpha, m) + \left(1 + \frac{1-m}{2}(\alpha+1)\right)\theta_k^2, \tag{1.17}$$

have a fourth-order of convergence.

Proof. In Section 1.1, we shown that the expansion (1.8) is true for parameter (1.6) of modified Popovski's family. Then the parameter τ_k given by (1.16) satisfies the fourth order convergence condition (1.9). Analogously, using (1.14) and (1.16) one can prove that the iteration (1.5) with τ_k given by (1.17) have a fourth order of convergence. $\square$

The iteration (1.5) with τ_k given by (1.16) and (1.17) we call the improved modifications of Popovski's and Laguerre's families respectively. The different modification Chebyshev's and Halley's methods were suggested by many authors (see for example [7,9,14,15,22]). As in [22] one can use the following approximations for w_k:

$$w_k \approx \frac{1}{2\theta_k}(f'(z_k) - f'(x_k))(y_k - x_k),\ z_k = x_k + \theta_k(y_k - x_k),\ 0 < \theta_k \leq 1. \tag{1.18}$$

$$\begin{aligned} w_k \approx (1+\frac{b}{2})f(y_k) + bf(x_k) - \frac{b}{2}f(z_k),\ z_k = x_k + f'(x_k)^{-1}f(x_k),\\ -2 \leq b \leq 0 \end{aligned} \tag{1.19}$$

in Popovski's and Laguerre's methods. But the approximation (1.4) is better than approximations (1.18) and (1.19) from the computational complexity of view.

Now we consider three-step iterative methods

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\ z_k &= x_k - \tau_k\frac{f(x_k)}{f'(x_k)},\\ x_{k+1} &= z_k - \alpha_k\frac{f(z_k)}{f'(x_k)}, \end{aligned} \tag{1.20}$$

where τ_k in (1.20) is given by (1.16) or (1.17), i.e., the first two steps in (1.20) is the same as improved modification of Popovski's or modified generalization of Laguerre's family of methods. According to Theorem 4 in [23], the iterative methods (1.20) has sixth and seventh order of convergence if α_k is given by

$$\alpha_k = 1 + 2\theta_k \tag{1.21}$$

and

$$\alpha_k = 1 + 2\theta_k + \Big(\frac{(1-2r)(1-3r)}{3(1-r)^2} + 1\Big)\theta_k^2 + \frac{f(z_k)}{f(y_k)} + O(f_k^3), \tag{1.22}$$

respectively, (see also, [19]).

2. Extension of modifications of Popovski's and Laguerre's family of methods to multidimensional case.

We consider the following nonlinear system of equations:

$$F(x) = 0,\quad x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n, \tag{2.1}$$

where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear and sufficiently Fréchet differentiable function in an open convex set D. Additionally, $F'(x)$ is continuous and nonsingular at x^*, where x^* is the simple and isolated root of equation (2.1). The solution of equation (2.1) cannot be computed exactly and often used iterative methods with different order convergence. Now, we extend modifications of Popovski's and Laguerre's family of methods to the system of nonlinear equations. We consider two-step iterative methods

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= x_k - T_k F'(x_k)^{-1}F(x_k), \ \ k = 0, 1, 2 \dots . \end{aligned} \tag{2.2}$$

The first and second steps in (2.2) are the same as improved modifications of Popovski's and Laguerre's family of methods, i.e., T_k in (2.2) is given by

$$T_k = \frac{(1-r)}{2\Theta_k}\left\{\left[\mathbf{1} - \frac{2r}{1-r}\Theta_k\right]^{1/r} - \mathbf{1}\right\} + \frac{2(2-r)}{3(1-r)}\Theta_k^2, \tag{2.3}$$

or

$$T_k = \frac{\alpha+1}{\alpha + (\mathbf{1} - m(\alpha+1)\Theta_k)^{1/m}} + \left(1 - \frac{1-m}{2}(\alpha+1)\right)\Theta_k^2, \tag{2.4}$$

where $\Theta_k = F(y_k)/F(x_k)$ and $\mathbf{1} = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. We here consider $\mathbb{R}^n$ with following operations of componentwise multiplication and division [6] of vectors $a = (a_1, a_2, \cdots, a_n)^T \in \mathbb{R}^n$ and $b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$:

$$\begin{aligned} & a \cdot b = (a_1 b_1, a_2 b_2, \cdots, a_n b_n)^T \in \mathbb{R}^n, \\ & \frac{a}{b} = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_n}{b_n}\right)^T \in \mathbb{R}^n. \\ & \sqrt[m]{a} = (\sqrt[m]{a_1}, \sqrt[m]{a_2}, \cdots, \sqrt[m]{a_n})^T \in \mathbb{R}^n, \ \ m > 0. \end{aligned} \tag{2.5}$$

The fourth-order convergence condition for method (2.2) is [21]

$$T_k = I + \vartheta_k + 2\vartheta_k^2 + d_k + O(h^3), \tag{2.6}$$

where

$$\begin{aligned} & \vartheta_k = \frac{1}{2}F'(x_k)^{-1}F''(x_k)\xi_k, \ \ d_k = -\frac{1}{6}F'(x_k)^{-1}F'''(x_k)\xi_k^2, \\ & \xi_k = F'(x_k)^{-1}F(x_k). \end{aligned} \tag{2.7}$$

Below, we give the equivalent formulation of (2.6) in term of $\Theta_k = \dfrac{F(y_k)}{F(x_k)}$. The condition (2.6) is equivalent to

$$T_k = \mathbf{1} + \Theta_k + 2\Theta_k^2 + O(h^3). \tag{2.8}$$

Proof. The Taylor expansion of $F(y_k)$ at the point x_k gives

$$F(y_k) = \frac{F''(x_k)}{2}\xi_k^2 - \frac{F'''(x_k)}{6}\xi_k^3 + O(h^4).$$

Thereby we have

$$F(x_k)^{-1}F(y_k) = (\vartheta_k + d_k)\xi_k + O(h^4).$$

Hence

$$\frac{F(x_k)^{-1}F(y_k)}{F(x_k)^{-1}F(x_k)} = \vartheta_k + d_k + O(h^3). \tag{2.9}$$

We denote by η_k the left-hand side of (2.9) i.e.,

$$\frac{F(x_k)^{-1}F(y_k)}{F(x_k)^{-1}F(x_k)} = \eta_k.$$

Then

$$F(x_k)^{-1}F(y_k) = F(x_k)^{-1}F(x_k)\eta_k \implies F(y_k) = F(x_k)\eta_k \implies \eta_k = \frac{F(y_k)}{F(x_k)} = \Theta_k.$$

From (2.9) it follows that

$$\Theta_k = \vartheta_k + d_k + O(h^3). \tag{2.10}$$

From (2.2) it clear that τ_k may be defined as vector, i.e., $\tau_k \in \mathbb{R}^n$. Thus using (2.10) one can formulate (2.6) in term of Θ_k as:

$$T_k = \mathbf{1} + \Theta_k + 2(\Theta_k - d_k)^2 + O(h^3) = \mathbf{1} + \Theta_k + 2\Theta_k^2 + O(h^3),$$

due to $\Theta_k = O(h)$ and $d_k = O(h^2)$. The converse is obvious due to (2.10). □

Using the (1.7) it is easy to show that τ_k given by (2.3) or (2.4) satisfies (2.8). So, the convergence order of (2.2) with (2.3) or (2.4) is equal to four. When $n = 1$ the condition (2.8) leads to (1.9).

Furthermore, we consider three-step iterative methods

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - T_kF'(x_k)^{-1}F(x_k), \\ x_{k+1} &= z_k - \alpha_kF'(x_k)^{-1}F(z_k), \end{aligned} \tag{2.11}$$

where T_k is given by (2.3) or (2.4). The order of convergence of iteration (2.11) equal to six, iff α_k satisfies

$$\alpha_k = \mathbf{1} + 2\Theta_k + O(h^2). \tag{2.12}$$

Proof. According to Theorem 3.1 in [21], the iteration (2.11) has order $p + 2$ iff

$$\alpha_k = I + 2\vartheta_k + O(h^2), \tag{2.13}$$

where ϑ_k is given by (2.7).

On the order hand, from last step of (2.11) it clear that α_k may also be defined as vector, as τ_k given by (2.8). The vector variant of (2.13) is (2.12).

Indeed, using Taylor expansion of $F(x_{k+1})$ at point z_k we obtain

$$F(x_{k+1}) = (F'(x_k) - F'(z_k)\alpha_k)F'(x_k)^{-1}F(z_k) + O(F(z_k)^2).$$

From this it clear that

$$F(x_{k+1}) = O(h^{p+2}),$$

iff

$$\alpha_k = F'(z_k)^{-1}F'(x_k) + O(h^2). \tag{2.14}$$

Using Taylor expansion of $F'(z_k)$ at point x_k, we have

$$F'(z_k) = F'(x_k)(I - 2\vartheta_kT_k) + O(h^2).$$

So

$$F'(z_k)^{-1} = (I - 2\vartheta_kT_k)^{-1}F'(x_k)^{-1} + O(h^2). \tag{2.15}$$

Substituting (2.15) into (2.14) we obtain

$$\alpha_k = (I - 2\vartheta_kT_k)^{-1} = \mathbf{1} + 2\vartheta_kT_k + O(h^2). \tag{2.16}$$

It is easy to show that

$$\vartheta_k = \Theta_k + O(h^2). \tag{2.17}$$

Using (2.8) and (2.17) in (2.16) we obtain (2.12). □

Analogously, proved seventh order of iteration (2.11) under

$$\alpha_k = \mathbf{1} + 2\Theta_k + \Big(\frac{(1-2r)(1-3r)}{3(1-r)^2} + 1\Big)\Theta_k^2 + \frac{F(z_k)}{F(y_k)}. \tag{2.18}$$

3. Numerical experiments

We determine which member of the presented families (1.5), (2.2) and (2.11) give better convergence from the numerical experiment and the numerical comparisons. Therefore, we consider several test problems, one of them are from real-life problem, e.g., Van der Waals problem of the state. We begin with the nonlinear function :

$$f_1(x) = x\exp(x^2) - \sin^2(x) + 3\cos(x) + 5,$$

with initial value $x_0 = -1.4$ towards the solution $x^* \approx -1.20764782$.

Van der Waal's equation [1]:

$$\Big(P + \frac{an^2}{V^2}\Big)(V - nb) = nRT,$$

where a and b are Van der Waal's constant. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V:

$$PV^3 - (nbP + nRT)V^2 + an^2V - an^2b = 0,$$

By considering the particular values of parameters, namely a and b, n, P and T, we can easily get the following nonlinear function:

$$f_2(x) = 0.986x^3 - 5.181x^2 + 9.067x - 5.289.$$

$f_2(x)$ has three zeros and our desired solution is $x^* \approx 1.92984624$. Additionally, we consider the initial value as $x_0 = 2$.

Consider the nonlinear system with $n = 20$ is given by

$$\begin{cases} x_{(i+1)}x_{(i)}^2 - 1 = 0, & i = 1, 2, \cdots, n-1, \\ x_{(1)}x_{(n)}^2 - 1 = 0. \end{cases}$$

A solution $x^* = (1, 1, \cdots, 1)^T$, initial value $x_0 = (1.25, 1.25, \cdots, 1.25)^T$. We choose a large system taken from (see [2], [7]):

$$\sum_{j=1, j\neq i}^{n} x_{(j)} - e^{x_{(i)}} = 0, \ 1 \le i \le n,$$

with initial guess $x_0 = (0.5, \cdots, 0.5)^T$. The results of comparisons for this problem are brought forward in Table 10. Finally, we consider the boundary value problem (see [2]):

$$u'' = \frac{u^3}{2} + 3u' - \frac{3}{2-x} + \frac{1}{2}, \ u(0) = 0, \ u(1) = 1.$$

We assume that the interval $[0, 1]$ is partitioned as below:

$$x_0 = 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1,\ x_{i+1} = x_i + h,\ h = 1/n.$$

Let us define $u_0 = u(x_0) = 0,\ \ u_1 = u(x_1), \ldots, u_n = u(x_n) = 1$. If we discretize the problem by using the numerical formulae for first and second derivatives

$$u_i' \approx \frac{u_{i-1} - u_{i+1}}{h^2},\ u_i'' \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},\ (i = 1, 2, 3, \cdots, n-1),$$

we get the following system of $n-1$ nonlinear equations in $n-1$ variables:

$$u_{i+1} - 2u_i + u_{i-1} - \frac{h^2}{2}u_i^3 - \tfrac{3}{2}h(u_{i+1} - u_{i-1}) + \frac{3}{2 - x_i}h^2 - \frac{h^2}{2} = 0,$$
$$i = 1, 2, \ldots, n-1.$$

We here consider $n = 21, 50, 100$ and solve this problem by selecting $u^{(0)} = (1, 1 \cdots, 1)^T$ as the initial value. The results of comparisons for this problem is displayed in Table 11. All computations are performed on the programming package Mathematica 12 using multiple-precision arithmetics. We have considered 1000 digits floating point arithmetics so as to minimize the round-off errors as much as possible. For each case, we are going to analyze the number of iterations k such that $|x_k - x_{k-1}| \leq 10^{-100}$.

The results of the numerical experiments can be found in Tables 1-6. These tables include the number of iterations k, absolute error in the consecutive iterations $|x_k - x_{k-1}|$, absolute residual error of the corresponding function $|f(x_k)|$ and computational order of convergence ρ_c. Here, the computational order of convergence ρ_c, defined in [12, 18], and is given by the formula

$$\rho_c \approx \frac{\ln(|x_k - x_{k+1}|/|x_{k-1} - x_k|)}{\ln(|x_{k-1} - x_k|/|x_{k-2} - x_{k-1}|)}.$$

where x_{k-2}, x_{k-1}, x_k are three consecutive approximations of iterations.

It is clear that these numerical results are in accordance with the theory. From Table 1, we observe that the iterative method (1.5) with (1.6) for $r = 2$ is better than other cases. The results corresponding to the same kind of experiments for (1.5) with (1.11) and parameters m, α can be found in Table 2. From Table 2, we see that the iterative method (1.5) with (1.11) for $m = 2,\ \alpha = 1$ is better than other cases.

Furthermore, from Tables 3 and 4, we can observe that the order of convergence of the method (1.5) with (1.16) and (1.17) is four. Tables 5 and 6 show that the convergence order of the method (1.20) with (1.16), (1.21) and (1.22) is equal to six and seven, respectively.

From Tables 7-9 we also see that the order of convergence of the methods (2.2) and (2.11) are four, six and seven, respectively. Thus, the order of convergence of the methods is successfully verified by numerical experiments in the multidimensional case. In this case, the order of convergence of the methods is computed by the formula given in [7].

In order to choose the best value of parameters and to corroborate our conclusion made in the numerical experiments, we also investigated basins of attraction of the methods.

Lastly, we can see that the choice $r = 2$ is the best one in most cases, both the basin of attractions and the numerical experiments. It can be noticed that methods (1.5) for $r = 2$ and $m = 2, \alpha = 1$ can coincides with each other.

For convenience, we denote our fourth-order method (2.2) with τ_k defined by (2.3) by $M^4_{(23)}(r)$, sixth and seventh-order methods (2.11) with α_k defined by (2.12) by $M^6_{(27)}(r)$ and α_k defined by (2.18) by $M^7_{(27)}(r)$, $r = -1, -2, 2, 3$. We compare the computational time of our fourth-order methods with method $M4$ [21] and method $M_{4,3}$ [16] and our sixth-order methods with method OS_6 [2] and method $M_{(6,2)}(\frac{1}{2},0)$ [7] and our seventh-order methods with method M_1^7 [17], respectively. The computational CPU time (in seconds) is reported in Tables 10, 11. From Tables 10, 11 it can be concluded that the proposed methods are more efficient and faster than the considered methods of the same order.

TABLE 1. *Convergence behavior of methods* (1.5) *with* (1.6) *on each* f_i

r	k	$\lvert x_k - x_{k-1}\rvert$	$\lvert f(x_k)\rvert$	ρ_c
			$f_1(x)$, $x_0 = -1.4$	
1/2	5	0.1050e-178	0.2132e-177	3.00
-2	5	0.3396e-216	0.6897e-215	3.00
2	4	0.7660e-248	0.1555e-246	4.00
			$f_2(x)$, $x_0 = 2$	
1/2	6	0.9244e-183	0.8992e-182	3.00
-2	5	0.3197e-172	0.7589e-171	3.00
2	4	0.7511e-212	0.2531e-212	4.00

TABLE 2. *Convergence behavior of methods* (1.5) *with* (1.11) *on each* f_i

m	α	k	$\lvert x_k - x_{k-1}\rvert$	$\lvert f(x_k)\rvert$	ρ_c
				$f_1(x)$, $x_0 = -1.4$	
2	1	4	0.7660e-248	0.1555e-248	4.00
3	1	5	0.4360e-194	0.1082e-193	3.00
4	2	5	0.1010e-138	0.2506e-138	3.00
				$f_2(x)$, $x_0 = 2$	
2	1	4	0.3125e-202	0.7412e-201	4.00
3	1	5	0.9125e-146	0.2356e-145	3.00
4	2	5	0.4311e-192	0.3145e-191	3.00

TABLE 3. *Convergence behavior of methods* (1.5) *with* (1.16) *on each* f_i

r	k	$\lvert x_k - x_{k-1}\rvert$	$\lvert f(x_k)\rvert$	ρ_c
			$f_1(x)$, $x_0 = -1.4$	
2	4	0.7660e-248	0.1555e-246	4.00
3	4	0.2743e-240	0.5571e-239	4.00
4	4	0.1798e-219	0.3652e-218	4.00
			$f_2(x)$, $x_0 = 2$	
2	4	0.9824e-241	0.2456e-240	4.00
3	4	0.4365e-203	0.4532e-202	4.00
4	4	0.2456e-213	0.1359e-212	4.00

TABLE 4. *Convergence behavior of methods* (1.5) *with* (1.17) *on each* f_i

m	α	k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$	ρ_c
				$f_1(x)$, $x_0 = -1.4$	
2	1	4	0.7660e-248	0.1555e-246	4.00
2	2	4	0.1305e-155	0.2651e-154	4.00
3	2	4	0.7608e-282	0.1545e-280	4.00
				$f_2(x)$, $x_0 = 2$	
2	1	4	0.6895e-266	0.1555e-265	4.00
2	2	4	0.1358e-226	0.2651e-225	4.00
3	2	4	0.8123e-219	0.1545e-218	4.00

TABLE 5. *Convergence behavior of methods* (1.20) *with* (1.16), (1.21).

r	k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$	ρ_c
			$f_1(x)$, $x_0 = -1.4$	
2	3	0.1421e-170	0.2887e-169	6.00
3	3	0.4829e-157	0.9806e-156	6.00
4	3	0.2286e-159	0.4642e-158	6.00
			$f_2(x)$, $x_0 = 2$	
2	3	0.1023e-241	0.3021e-240	6.00
3	3	0.2314e-221	0.1245e-220	6.00
4	3	0.9872e-199	0.9874e-198	6.00

TABLE 6. *Convergence behavior of methods* (1.20) *with* (1.16), (1.22).

r	k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$	ρ_c
			$f_1(x)$, $x_0 = -1.4$	
2	3	0.2703e-256	0.5489e-255	7.00
3	3	0.1796e-231	0.3647e-230	7.00
4	3	0.1430e-241	0.2905e-240	7.00
			$f_2(x)$, $x_0 = 2$	
2	3	0.4762e-253	0.2456e-252	7.00
3	3	0.1546e-223	0.4532e-223	7.00
4	3	0.3456e-207	0.1359e-206	7.00

TABLE 7. *Convergence behavior of methods* (2.2) *for example* (3).

r	k	$\\|x_k - x_{k-1}\\|$	$\\|f(x_k)\\|$	ρ_c
			parameter (2.3)	
2	5	0.3467e-202	0.2771e-200	4.00
-2	5	0.0439e-139	0.4944e-136	3.99
3	5	0.2873e-171	0.1329e-169	3.99
(m, α)	k	$\\|x_k - x_{k-1}\\|$	$\\|f(x_k)\\|$	ρ_c
			parameter (2.4)	
$(2, 1)$	5	0.3467e-202	0.2771e-200	4.00
$(1, 2)$	5	0.7321e-136	0.2993e-134	3.99
$(3, 2)$	5	0.0532e-153	0.9328e-151	3.99

TABLE 8. *Convergence behavior of methods* (2.11) *for example 3.*

r	k	$\\|x_k - x_{k-1}\\|$	$\\|f(x_k)\\|$	ρ_c
			parameters (2.3) (2.12)	
2	4	0.8303e-806	0.9467e-804	6.00
-1	4	0.1582e-896	0.7956-901	5.99
-2	4	0.7485e-623	0.1044e-621	5.99
3	4	0.7012e-718	0.1771e-716	5.99
(m, α)	k	$\\|x_k - x_{k-1}\\|$	$\\|f(x_k)\\|$	ρ_c
			parameters (2.4) (2.12)	
$(2, 1)$	4	0.8303e-806	0.9467e-804	6.00
$(1, 2)$	4	0.1991e-615	0.1781e-613	5.99
$(3, 2)$	4	0.5411e-701	0.1456e-699	5.99

TABLE 9. *Convergence behavior of methods (2.11) for example 3.*

r	k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$	ρ_c
			parameters (2.3)	(2.18)
2	4	0.1245e-1349	0.1456e-1349	7.00
-2	4	0.3651e-1235	0.1001e-1235	6.99
3	4	0.2563e-1249	0.1999e-1249	6.99
(m, α)	k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$	ρ_c
			parameters (2.4)	(2.18)
$(2, 1)$	4	0.5789e-1349	0.4659e-1349	7.00
$(1, 2)$	4	0.3281e-1211	0.2135e-1211	6.99
$(3, 2)$	4	0.1231e-1198	0.2451e-1198	6.99

TABLE 10. *The computational time for Example 3.*

Methods	ρ	$n = 20$	$n = 50$	$n = 100$
$M^4_{(23)}(-1)$	5*4	2.79	17.15	63.71
$M^4_{(23)}(-2)$		4.64	20.37	106.27
$M^4_{(23)}(2)$		4.60	17.70	64.01
$M^4_{(23)}(3)$		4.64	17.87	65.05
$M4$ [21]		4.25	20.21	70.25
$M_{4,3}$ [16]		4.96	22.71	78.91
$M^6_{(27)}(-1)$	5*6	3.39	16.00	161.82
$M^6_{(27)}(-2)$		4.81	20.72	211.42
$M^6_{(27)}(2)$		4.66	23.54	233.61
$M^6_{(27)}(3)$		3.93	15.16	60.42
OS_6 [2]		9.17	91.84	321.71
$M_{(6,2)}(\frac{1}{2}, 0)$ [7]		6.89	59.66	198.21
$M^7_{(27)}(-1)$	5*7	11.67	52.81	157.20
$M^7_{(27)}(-2)$		29.05	90.10	389.30
$M^7_{(27)}(2)$		32.62	99.41	319.62
$M^7_{(27)}(3)$		28.97	87.51	262.23
M^7_1 [17]		30.88	99.77	303.11

TABLE 11. *The computational time for Example 3*

Methods	Order	$n = 20$	$n = 50$	$n = 100$
$M^4_{(23)}(-1)$	5*4	3.06	24.06	150.76
$M^4_{(23)}(-2)$		4.17	27.74	157.37
$M^4_{(23)}(2)$		4.08	27.31	156.32
$M^4_{(23)}(3)$		4.01	27.08	156.04
$M4$ [21]		5.46	33.65	221.31
$M_{4,3}$ [16]		6.11	42.25	285.71
$M^6_{(27)}(-1)$	5*6	3.81	31.67	212.07
$M^6_{(27)}(-2)$		4.59	35.02	224.11
$M^6_{(27)}(2)$		4.76	34.69	218.56
$M^6_{(27)}(3)$		4.61	33.94	212.09
OS_6 [2]		13.79	187.54	501.78
$M_{(6,2)}(\frac{1}{2}, 0)$ [7]		12.56	147.14	421.74
$M^7_{(27)}(-1)$	4*7	4.00	33.62	212.24
$M^7_{(27)}(-2)$		5.50	39.91	242.41
$M^7_{(27)}(2)$		4.82	36.52	235.11
$M^7_{(27)}(3)$		4.57	35.72	216.40
M^7_1 [17]		10.94	88.56	519.76

4. Conclusion

In this work, we presented two modifications of Popovski's and Laguerre's families of methods with the fourth, sixth, and seventh order of convergence in order to solve nonlinear problems. The main contributions of this paper are:

- we obtain simple modifications of considered methods free from second derivative
- we improve the convergence order of Popovski's and Laguerre's methods
- we propose the extension of modifications to solve nonlinear systems and hence we enlarge the domain of applications of these methods.
- To illustrate efficiency and accuracy, the numerical experiments are carried out for different systems of nonlinear equations.

We can conclude that the choices of parameters $r = 2$ and $m = 2, \alpha = 1$ are the best choice in most cases both the basin of attractions and the numerical experiments. Note that all members of modifications of Popovski's and Laguerre's families of methods have no extraneous roots. Finally, we compared the performance of presented methods to some existing methods and found that our methods remarkably reduce the computing time in calculations.

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