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Gordon Growth Model with Vector Autoregressive Process

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Abstract. In this study, we introduce a Gordon's dividend discount model, based on Vector Auto Regressive Process (VAR). We provide two Propositions, which are related to the generic Gordon growth model and the Gordon growth model, which is based on the VAR process.

1. Introduction

Dividend discount models (DDMs), first introduced by Williams (1938), are a popular tool for stock valuation. After Williams (1938) initial model, Gordon and Shapiro (1956) introduced a more sophisticated one, in what dividends change by a defined (deterministic) growth rate. If we assume that a rm will not default in the future, then the basic idea of all DDMs is that the market price of a stock equals the sum of the stocks next period price and dividends discounted at a risk-adjusted rate, known as a required rate of return, see, e.g., Brealey, Myers, and Marcus (2020). By their very nature, DDM approaches are best applicable to companies paying regular cash dividends. For a DDM with default risk, we refer to Battulga, Jacob, Altangerel, and Horsch (2022). As the outcome of DDMs depends crucially on dividend forecasts, most research in the last few decades has been around the proper estimations of dividend development. To obtain higher order moments of stock prices, Battulga et al. (2022) used non-homogeneous Poisson process for dividends. By using the DDM and regime switching process, Battulga (2022) obtained pricing and hedging formulas for European options and equity linked life insurance products. To model the required rate of return on stock, Battulga (2023) applied a three regime model. The result of the paper reveals that the regime switching model is good t for the required rate of return. A reviews of some existing deterministic and stochastic DDMs, which model future dividends can be found in DAmico and De Blasis (2020) and Battulga et al. (2022).

The rest of the paper is organized as follows: In Section 2, for a DDM, whose dividend is modeled by the popular Gordon growth model, we obtain its theoretical price formulas. Also, we provide a proposition, which deals with two representations of theoretical stock prices. Section 3 is dedicated to the Gordon growth model, which is based on the VAR process. In this Section, we provide existence conditions on stock prices and their second-order moments. Finally, Section 4 concludes the study.

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2. Gordon Growth Model

In this paper, we assume that there are m companies and the companies will not default in the future. As mentioned before the basic idea of all DDMs is that the market price of a stock equals the sum of the stocks next period price and dividends discounted at the required rate of return. Therefore, for successive prices of i-th company, the following relation holds

$$P_{i,t} = (1 + k_{i,t})P_{i,t-1} - d_{i,t}, \quad i = 1, \dots, m, t = 1, 2, \dots \tag{2.1}$$

where $k_{i,t}$ is the required rate of return on stock, $P_{i,t}$ is the stock price, and $d_{i,t}$ is the dividend, respectively,at time t of i-th company. In vector form, the above equation is written by

$$P_t = (i_m + k_t) \odot P_{t-1} - d_t, \quad t = 1, 2, \dots \tag{2.2}$$

where $\odot$ is the Hadamard's element-wise product of two vectors, $i_m = (1, \dots, 1)'$ is an $(m \times 1)$ vector, consisting of 1, $k_t := (k_{1,t}, \dots, k_{m,t})'$ is a vector of the required rate of returns on stocks, $P_t = (P_{1,t}, \dots, P_{m,t})'$ price vector and $d_t = (d_{1,t}, \dots, d_{m,t})'$ is a dividend vector at time t of the companies. On the other hand, according to the Gordon growth model, the successive dividends of companies are modeled by

$$d_t = (i_m + k_t) \odot d_{t-1}, \quad t = 1, 2, \dots \tag{2.3}$$

Where $g_t := (g_{1,t}, \dots, g_{m,t})'$ is a vector of dividend growth rate at time t of the companies. In this paper, we assume that $k_{i,t} > -1$ and $g_{i,y}$ for all $i = 1, \dots, m$ and $t = 1, 2, \dots$.

From equation (2.2) and (2.3), the required rate of returns and dividends of the companies are obtained by

$$k_t = (P_t + d_t) \oslash P_{t-1} - i_m \tag{2.4}$$

$$g_t = d_t \oslash d_{t-1} - i_m \tag{2.5}$$

for all $t = 1, 2, \dots$ with the element-wise division of two vectors, $\oslash$ notation.

Furthermore, along $t = 1, 2, \dots$ and $r = 0, 1, \dots$, we can calculate two formulas for the price vector at time $t + r$ of the companies

$$P_{t+r} = \sum_{q=1}^{\infty} \left(\prod_{j=r+1}^{r+q} (i_m + g_{t+j}) \oslash (i_m + k_{t+j}) \right) \odot \prod_{j=1}^{r} (i_m + g_{t+j}) \odot d_t \tag{2.6}$$

$$P_{t+r} = \prod_{j=1}^{r} (i_m + k_{t+j}) \odot P_t - \sum_{q=1}^{r} \left(\prod_{j=r+1}^{r+q} (i_m + k_{t+j}) \odot (i_m + g_{t+j}) \right) \odot d_t \tag{2.7}$$

where $\prod_{j=1}^{q} o_j = o_1 \odot o_2 \cdot \odot o_q$ is an element-wise product of the vectors $o_1, o_2, \dots, o_q$ with convention $\sum_{j=q}^{q-1} o_j = 0$ and $\prod_{j=q}^{q-1} o_j = i_m$. Consequently, we have two equivalent equations at forecast origin t for future price vector at time $t + r$. The first one depends on infinite number of the future dividend growth rates and required rate of returns and dividend vector at time t, while second one depends on finite number of the future dividend growth rates and required rate of returns and dividend vector and price vector at t.

Also, one may write the above DDM equation in the following form

$$P_{i,t} = \exp\{\tilde{k}_{i,t}\} P_{i,t-1} - d_{i,t} \text{ for } i = 1, \dots, m, t = 1, 2, \dots, \tag{2.8}$$

where $\tilde{k}_{i,t} := ln(1 + k_{i,t})$ is a log required rate of return on stock at time t of i-th company. From the equation (2.1), vector form of the price vector can be rewritten as

$$P_t = \exp\{\tilde{k}_t\}P_{t-1} - d_t \text{ for } t = 1, 2, \dots, \tag{2.9}$$

where $\tilde{k}_t := (\tilde{k}_{1,t}, \dots, \tilde{k}_{m,t})'$ is $(m \times 1)$ vector with exponent term definition of $\exp\{\tilde{k}_t\} := (\exp\{\tilde{k}_{1,t}\}, \dots, \exp\{\tilde{k}_{m,t}\})'$.

From the equation (2.9), the log required rate of return vector at time t is rewritten as

$$\tilde{k}_t = \ln\big((P_t + d_t) \oslash P_{t-1}\big), \quad t = 1, 2, \dots, \tag{2.10}$$

For the equation (2.3), rewritten as a following log form,

$$\ln(d_t \oslash d_{t-1}) = \tilde{d}_t - \tilde{d}_{t-1} = \tilde{g}_t, t = 1, 2, \dots, \tag{2.11}$$

where $\tilde{d}_t := (\tilde{d}_{1,t}, \dots, \tilde{d}_{m,t})'$ is a log dividend vector at time t with $\tilde{d}_{i,t} := \ln(d_{i,t})$ and $\tilde{g}_t := (\tilde{g}_{1,t}, \dots, \tilde{g}_{m,t})'$ is a log dividend growth rate vector at time t with $\tilde{g}_{i,t} := \ln(1 + g_{i,t})$. Thus, in therms of the log dividend growth rates, log dividend at time $t + r$ is written by

$$\tilde{d}_{t+r} = \tilde{d}_t + \sum_{j=1}^{r} \tilde{g}_{t+j}, \quad r \geq 0 \tag{2.12}$$

Consequently,reformulating the equations (2.6, 2.7, 2.10 and 2.12), the price vector at time $t + r$ are given by the following equation

$$P_{t+r} = \sum_{q=1}^{\infty} \exp\left\{\sum_{j=1}^{r} \tilde{g}_{t+j} + \sum_{j=r+1}^{r+q} \left(\tilde{g}_{t+j} - \tilde{k}_{t+j}\right)\right\} \tag{2.13}$$

$$P_{t+r} = \exp\left\{\sum_{j=1}^{r} \tilde{k}_{t+j}\right\} \odot P_t - \sum_{q=1}^{r} \exp\left\{\sum_{j=q+1}^{r} \tilde{k}_{t+j} + \sum_{j=1}^{q} \tilde{k}_{t+j}\right\} \odot d_t \tag{2.14}$$

Again, we have two equivalent equations at forecast origin t for future price vector at time $t + r$. The first one depends on infinite number of the future log dividend growth rates and log of required rate of returns and dividend vector at time t, while second one depends on finite number of the future log dividend growth rates and log required rate of returns and dividend vector and price vector at time t. Then, a question arise that which one is dominant in some sense? The following Proposition may answer the question.

PROPOSITION 1. *1. Let $\mathcal{F}$ and $\mathcal{G}$ be σ–field such that $\mathcal{F} \subset \mathcal{G}$ and X is a square integrable random variable, defined on a probability space $(\Omega, \mathcal{H}_T, \mathbb{P})$. Then, conditional on both of the σ–fields $\mathcal{F}$ and $\mathcal{G}$, the mean squared distance between the random variable X and a conditional expectation $\mathbb{E}(X|\mathcal{G})$ is less than or equal to the distance between the random variable X and a conditional expectation $\mathbb{E}(X|\mathcal{F})$. Also, expectations of the conditional expectations $\mathbb{E}(X|\mathcal{G})$ and $\mathbb{E}(X|\mathcal{F})$ are equal to $E(X)$.*

Proof.

$$\begin{aligned}\mathrm{Var}(X|\mathcal{G}) =&\mathbb{E}\big((X-\mathbb{E}(X|\mathcal{G}))^2|\mathcal{G})\big) = \mathbb{E}\big((X-\mathbb{E}(X|\mathcal{F}))^2|\mathcal{G})\big)\\ &-2\mathbb{E}\big((X-\mathbb{E}(X|\mathcal{F}))(\mathbb{E}(X|\mathcal{G})-\mathbb{E}(X|\mathcal{F}))|\mathcal{G}\big)+\mathbb{E}\big((\mathbb{E}(X|\mathcal{G})-\mathbb{E}(X|\mathcal{F}))^2|\mathcal{G}\big)\end{aligned}$$

Since $\mathbb{E}(X|\mathcal{G})-\mathbb{E}(X|\mathcal{F})$ is measurable with respect to the σ-field $\mathcal{G}_t$, we have

$$\mathrm{Var}(X|\mathcal{G}) = \mathbb{E}\big((X-\mathbb{E}(X|\mathcal{F}))^2|\mathcal{G}\big)-\big((\mathbb{E}(X|\mathcal{G})-\mathbb{E}(X|\mathcal{F})\big)^2$$

Consequently,

$$\mathbb{E}\big((X-\mathbb{E}(X|\mathcal{G}))^2|\mathcal{G}\big)\leq \mathbb{E}\big((X-\mathbb{E}(X|\mathcal{F}))^2|\mathcal{G}\big)$$

If we take conditional expectation from the above inequality with respect to the σ–field $\mathcal{F}$, one gets

$$\mathbb{E}\big((X-\mathbb{E}(X|\mathcal{G}))^2|\mathcal{F}\big)\leq \mathbb{E}\big((X-\mathbb{E}(X|\mathcal{F}))^2|\mathcal{F}\big)$$

Thus, conditional on both of the σ–field $\mathcal{F}$ and $\mathcal{G}$, the mean squared distance between the random variable X and a conditional expectation $\mathbb{E}(X|\mathcal{G})$ is less than or equal the distance between the random variable X and a conditional expectation $\mathbb{E}(X|\mathcal{F})$. Finally $\mathbb{E}(X)=\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\big)=\mathbb{E}\big(\mathbb{E}(X|\mathcal{F})\big)$. That completes the proof of the Proposition. □

The proposition tells us that the two forecasts $\mathbb{E}(X|\mathcal{G})$ and $\mathbb{E}(X|\mathcal{F})$ have the same mean, but the forecast $\mathbb{E}(X|\mathcal{G})$ is closer to the random variable X than the forecast $\mathbb{E}(X|\mathcal{F})$ in the sense of mean squared distance. As a result, because for public companies, their prices are observed, to find future theoretical price s(forecast) one may use equations (2.7) and (2.14), that is, the best theoretical price at time $t+r$, which starts at the forecast origin t is given by

$$\begin{aligned}&\mathbb{E}[P_{t+r}|\mathcal{G}_t]\\ &=\mathbb{E}\left[\prod_{j=1}^{r}(i_m+k_{t+j})\bigg|\mathcal{G}_t\right]\odot P_t-\sum_{q=1}^{r}\mathbb{E}\left[\left(\prod_{j=r+1}^{r+q}(i_m+k_{t+j})\odot(i_m+g_{t+j})\right)\bigg|\mathcal{G}_t\right]\odot d_t\\ &=\mathbb{E}\left[\exp\left\{\sum_{j=1}^{r}\tilde{k}_{t+j}\right\}\bigg|\mathcal{G}_t\right]\odot P_t-\sum_{q=1}^{r}\mathbb{E}\left[\exp\left\{\sum_{j=q+1}^{r}\tilde{k}_{t+j}+\sum_{j=1}^{q}\tilde{k}_{t+j}\right\}\bigg|\mathcal{G}_t\right]\odot d_t \quad (2.15)\end{aligned}$$

where $\mathcal{G}_t$ is is the information from both the dividends and the prices up to time t. However, equation (2.6) and (2.13) still can be used to determine theoretical price at time t of the companies, that is,

$$\begin{aligned}\mathbb{E}[P_t|\mathcal{F}_t]&=\sum_{q=1}^{\infty}\mathbb{E}\left[\left(\prod_{j=1}^{q}(i_m+g_{t+j})\oslash(i_m+k_{t+j})\right)\bigg|\mathcal{F}_t\right]\odot d_t\\ &=\sum_{q=1}^{\infty}\mathbb{E}\left[\exp\left\{\sum_{j=1}^{q}\Big(\tilde{g}_{t+j}-\tilde{k}_{t+j}\Big)\right\}\bigg|\mathcal{F}_t\right]\odot d_t \quad (2.16)\end{aligned}$$

where we used the monotone convergence theorem and $\mathcal{F}_t$ is the information from the dividends, but not from the prices, up to time t and such that $\mathcal{F}_t\subset\mathcal{G}_t$.

3. Simple VAR(p) process

We assume that the log required rate of return vector k_t and the log of dividend growth rate vector, d_t fort the first m and next m components of the Bayesian VAR(p) process with order p and dimension $2m$ respectively at time t. Let's denote the dimension of the Bayesian VAR(p) process by n, i.e., $n := 2m$. By applying the Monte-Carlo methods and equations (2.6), (2.7), (2.13) and (2.14), one obtains theoretical value of the prices and distributions of the companies. Henceforth, we consider the simple VAR process y_t, given by the following equation

$$y_t = v + A_1 y_{t-1} + \cdots + +A_p y_{t-p} + \xi_t \tag{3.1}$$

where $y_t = (y_{1,t}, \dots, y_{n,t})^T$ is an $(n \times 1)$ vector of endogenous variables, $v = (v_1, \dots, v_n)'$ is a $(n \times 1)$ Intercept vector, for $i = 1, \dots, p$, A_i are $(n \times n)$ coefficient matrices and $\xi_t = (\xi_{1,t}, \dots, \xi_{n,t})'$ is an $(n \times 1)$ Gaussian white noise process with zero mean vector and positive defined covariance matrix Σ.

In order to define a distribution of the VAR(p) process y_t, let us write the VAR(p) process in VAR(1) form

$$y_t^* = v^* + A^* y_{t-1}^* + \xi_t^* \tag{3.2}$$

Where $y_t^* := (y_t', \dots, y_{t-p+1}')$ is an $(np \times 1)$-dimensional process, $v^* := (v', 0, \dots, 0)'$ is and $(np \times 1)$ intercept vector, $\xi_t^* := (\xi_t', 0, \dots, 0)'$ is a $(np \times 1)$ white noise process and

$$A^* := \begin{bmatrix} A_1 & \dots & A_{p-1} & A_p \\ I_m & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I_m & 0 \end{bmatrix}$$

is an $(np \times np)$ matrix and we assume that the VAR(p) process is stable, which means the modulus of all eigenvalues of the matrix A^* is less than one (Lütkepohl (2005)). By repeating equation (3.2), one finds that for $j > 0$,

$$y_{t+j}^* = \sum_{q=1}^{j} A^{j-p} v^* + (A^*)^j y_t^* + \sum_{q=1}^{j} A^{j-q} \xi_{t+q}^*$$

Let us define an $(n \times np)$ matrix $J := [I_n, 0, \cdots, 0]$ with $0 := 0_{n\times n}$ matrix. The matrix can be used to extract the VAR(p) process y_t from the VAR(1) process y_t^*, i.e., $y_t = Jy_t^*$. Then since $J'Jv^* = v$ and $J'J\xi_{t+i}^* = \xi_{t+i}$, we have that for $j > 0$, with notation $\Phi_q = J(A^*)^q J'$,

$$y_{t+j} = \sum_{q=1}^{j} \Phi_{j-p} v + J(A^*)^j y_t^* + \sum_{q=1}^{j} \Phi_{j-q} \xi_{t+q} \tag{3.3}$$

The matrix Φ_q can be used to the impulse response analysis. From MA(∞) representation of the VAR(p) process, it can shown that

$$\frac{\partial y_{t+q}}{\partial \xi_t'} = \Phi_q \tag{3.4}$$

The matrix Φ_q has the following explanation that a (i,j)–th element of the matrix Φ_q, identifies the consequences of a one–unit increase in the j–th variable's innovation at date t for the value of the i–th variable at time $t+q$, holding all other innovations at all dates constant, see Hamilton (1994) and Lütkepohl (2005). Let e_i be an $(m \times 1)$ unit vector, whose i–th element is 1 and others are zero, $J_k := [I_m : 0 : 0]$ be an $(m \times n)$ matrix, whose first block is I_m and others are zero and $J_g := [0 : I_m : 0]$ be an $(m \times n)$ matrix, whose second block is I_m and others are zero, where I_m is an $(m \times m)$ identity matrix. The matrices can be used to extract the log required rate of return vector and the log dividend growth rate vector from the VAR(p) process, i.e., $\tilde{k}_t = J_k y_t$ and $\tilde{g}_t = J_g y_t$. Then, it follows from equations (2.14) and (3.4) that

$$\frac{\partial P_{t+r}}{\partial \xi_t'} = [a_1' : \cdots : a_m']'$$

where for each $i = 1, \dots, m$ $(1 \times n)$ vector a_i is given by

$$\begin{aligned} a_i = \exp\Bigg\{\sum_{j=1}^r e_i'\tilde{k}_{t+j}\Bigg\}P_{i,t}\Bigg(e_i'J_k\sum_{j=1}^r \Phi_j\Bigg) \\ -\sum_{q=1}^r \exp\Bigg\{\sum_{j=q+1}^r e_i'\tilde{k}_{t+j} + \sum_{j=1}^q e_i'\tilde{g}_{t+j}\Bigg\}d_{i,t}\Bigg(e_i'J_k\sum_{j=q+1}^r \Phi_j + e_i'J_g\sum_{j=1}^q \Phi_j\Bigg) \end{aligned}$$

It follows from the equation (3.3) that conditional on the information $\mathcal{F}_t$, expectation and covariance of the process y_{t+j} are given by

$$\mathbb{E}(y_{t+j}|\mathcal{F}_t) = \sum_{q=1}^j \Phi_{j-q}v + J(A^*)^j y_t^*,\ j > 0 \tag{3.5}$$

and

$$\mathrm{Cov}(y_{t+j_1}, y_{t+j_2}|\mathcal{F}_t) = \sum_{q=1}^{j_1\wedge j_2} \Phi_{j_1-q}\Sigma\Phi_{j_2-q}',\ j_1, j_2 > 0 \tag{3.6}$$

where the information equals $\mathcal{F}_t := \sigma(\tilde{d}_0, y_0, \dots, y_t)$. Furthermore, as $\xi_t \sim \mathcal{N}(0, \Sigma)$, using the equation (2.16) the theoretical price at time t of i-th company equals

$$\mathbb{E}(e_i'P_t|\mathcal{F}_t) = \sum_{q=1}^\infty \exp\Bigg\{e_i'\tilde{d}_t + \mathbb{E}(e_i'z_q|\mathcal{F}_t) + \frac{1}{2}\mathcal{D}\big[\mathrm{Var}(e_i'z_q|\mathcal{F}_t)\big]\Bigg\} \tag{3.7}$$

where $z_q := J_{g,k}\sum_{j=1}^q y_{t+j}$ is $(m \times 1)$ vector, $J_{g,k} := J_g - J_k$ is and $m \times m$ difference matrix and for a generic $m \times m$ matrix $\mathcal{O}$, $\mathcal{D}[\mathcal{O}]$ is a $(m \times 1)$ vector that consists of diagonal elements of the matrix). Also, by monotone convergence theorem, a mixed moment of stock prices at time t of i_1-th and i_2-th companies is evaluated by

$$\begin{aligned} \mathbb{E}(e_{i_1}'P_tP_t'e_{i_2}|\mathcal{F}_t) = \sum_{q_1=1}^\infty\sum_{q_2=1}^\infty \exp\Bigg\{e_{i_1}'\tilde{d}_t + e_{i_2}'\tilde{d}_t \\ + \mathbb{E}(e_{i_1}'z_{q_1} + e_{i_2}'z_{q_2}|\mathcal{F}_t) + \frac{1}{2}\mathcal{D}\big[\mathrm{Var}(e_{i_1}'z_{q_1} + e_{i_2}'z_{q_2}|\mathcal{F}_t)\big]\Bigg\} \end{aligned} \tag{3.8}$$

In the following proposition, we give sufficient conditions of convergences of series (3.7) and (3.8).

PROPOSITION 2. *Let all eigenvalues of the matrix A^* be different and their modulus are less than one (y_t is stable). Then, we have that*

(i) *if following inequality holds, then series (3.7) is convergent*

$$e_i'J_{g,k}\mu + e_i'J_{g,k}\left(\frac{1}{2}\Gamma_0 + \Gamma\right)J_{g,k}'e_i < 0 \tag{3.9}$$

(ii) *and if the following inequality holds, then series (3.8) is convergent*

$$\max\Big\{e_{i_1}'J_{g,k}\mu + e_{i_1}'J_{g,k}\big(\Gamma_0 + 2\Gamma\big)J_{g,k}'e_{i_1}, e_{i_2}'J_{g,k}\mu + e_{i_2}'J_{g,k}\big(\Gamma_0 + 2\Gamma\big)J_{g,k}'e_{i_2}\Big\} \tag{3.10}$$

where $\mu := \big[I - A_1 - \dots - A_p\big]^{-1}v$ *is a mean and* $\Gamma_0 := \sum_{i=0}^{\infty}\Phi_i\Sigma\Phi_i'$ *is a covariance matrix of the process* y_t *and*

$$\Gamma := \sum_{j_1=1}^{\infty}\sum_{j_2=j_1}^{\infty}\Phi_{j_2}\Sigma\Phi_{j_1-1}'$$

Proof. (i) Let us consider q-th term of the series (3.7), namely,

$$\hat{s}_{i,q} := e_i'\exp\left\{\tilde{d}_t + J_{g,k}\sum_{j=1}^{q}\mathbb{E}(y_{t+j}|\mathcal{F}_t) + \frac{1}{2}\mathcal{D}\left[\operatorname{Var}\left(J_{g,k}\sum_{j=1}^{q}y_{t+j}\Big|\mathcal{F}_t\right)\right]\right\}$$

Then, according to the ratio test of a series, a ratio of successive terms of the series equals

$$\begin{aligned}\frac{\hat{s}_{i,q+1}}{\hat{s}_{i,q}} = \exp\Bigg\{&e_i'J_{g,k}\mathbb{E}(y_{t+q+1}|\mathcal{F}_t) + \frac{1}{2}e_i'\mathcal{D}\big[\operatorname{Var}(J_{g,k}y_{t+q+1}|\mathcal{F}_t)\big]\\ &+ e_i'\mathcal{D}\left[\operatorname{Cov}\left(J_{g,k}y_{t+q+1}, J_{g,k}\sum_{j=1}^{q}y_{t+j}\Big|\mathcal{F}_t\right)\right]\Bigg\}\end{aligned} \tag{3.11}$$

By equations (3.5) and (3.6) and the fact that y_t is a stable process, for the first line of the above equation, it holds

$$\lim_{q\to\infty}\mathbb{E}(y_{t+q+1}|\mathcal{F}_t) = \mu$$

and

$$\lim_{q\to\infty}\operatorname{Var}(y_{t+q+1}|\mathcal{F}_t) = \sum_{q=0}^{\infty}\Phi_q\Sigma\Phi_q' = \Gamma_0$$

For the second line of equation (3.11), due to equation (3.6), we have

$$\operatorname{Cov}\left(y_{t+q+1}, \sum_{j=1}^{q}y_{t+j}\Big|\mathcal{F}_t\right) = \sum_{j_1=1}^{q}\sum_{j_2=j_1}^{q}\Phi_{j_2}\Sigma\Phi_{j_1-1}'$$

Since all eigenvalues are different, the matrix $(A^*)^j$ can be represented by $(A^*)^j = C\Lambda^jC^{-1}$, where the matrix C consists of eigenvectors of the matrix A^* and Λ is a diagonal matrix whose elements are eigenvalues of the matrix A^*. Consequently, since for generic vector $o_1, o_2 \in \mathbb{R}^n$ and matrix $\mathcal{O} \in \mathbb{R}^{n\times n}$, $\operatorname{diag}\{o_1\}\mathcal{O}\operatorname{diag}\{o_2\} = \mathcal{O}\odot(o_1o_2')$ and $\Phi_j = J(A^*)^JJ'$, it holds

$$\sum_{j_1=1}^{q}\sum_{j_2=j_1}^{q}\Phi_{j_2}\Sigma\Phi_{j_1-1}' = \sum_{j_1=1}^{q}\sum_{j_2=j_1}^{q}JC\Big(C^{-1}J'\Sigma J(C')^{-1}\odot(d_1d_2')\Big)C'J' \tag{3.12}$$

where $d_1 := \sum_{j_2=j_1}^{q} \mathcal{D}\big[\Lambda^{j_2}\big]$ and $d_2 := \mathcal{D}\big[\Lambda^{j_1-1}\big]$. Therefore, because VAR(p) process is stable, a generic (α,β)-th element of the matrix $\sum_{j_1=1}^{q} d_1 d_2'$ converges to

$$\lim_{q\to\infty}\left[\sum_{j_1=1}^{q}\sum_{j_2=j_1}^{q}\lambda_\alpha^{j_2}\lambda_\beta^{j_1-1}\right]=\frac{\lambda_\alpha}{(1-\lambda_\alpha)(1-\lambda_\beta)}$$

Therefore, the matrix given in equation (3.12) has a finite limit and we denote it by Γ. As a result, by the ratio test of a series, if condition (3.9) holds, the series (3.7) is convergent.

(ii) According to the Cauchy–Schwarz inequality, we get that

$$\begin{aligned}\mathbb{E}\big(e_{i_1}'P_tP_t'e_{i_2}|\mathcal{F}_t\big) &\leq \sqrt{\mathbb{E}\big(e_{i_1}'P_tP_t'e_{i_1}|\mathcal{F}_t\big)\mathbb{E}\big(e_{i_2}'P_tP_t'e_{i_2}|\mathcal{F}_t\big)}\\ &= \sqrt{\mathbb{E}\big((e_{i_1}'P_t)^2|\mathcal{F}_t\big)\mathbb{E}\big((e_{i_2}'P_t)^2|\mathcal{F}_t\big)}\end{aligned}$$

where for $k=1,2$,

$$\mathbb{E}\big((e_{i_k}'P_t)^2|\mathcal{F}_t\big) := \sum_{q_k=1}^{\infty}\exp\left\{2\Big(e_{i_k}'\tilde{d}_t+\mathbb{E}\big(e_{i_k}'z_{q_k}|\mathcal{F}_t\big)+\mathcal{D}\big[\operatorname{Var}\big(e_{i_k}'z_{q_k}|\mathcal{F}_t\big)\big]\Big)\right\}$$

As a result, by repeating the idea above, one obtains equation (3.10). That completes the proof. □

It is worth mentioning that if VAR(p) process, given by equation (3.1) is cointegrated, then by using the Granger representation theorem (see Hansen (2005)), it can be shown that the series (3.7) is diverging, that is, the theoretical stock price of i–th company does not exist. Consequently, higher order moments of stock prices also do not exist. Finally, to use the Gordon growth model, based on the VAR process, one needs parameter estimation. Parameter estimation method can found in Hamilton (1994) and Lütkepohl (2005). One may be extend the results in this section to Bayesian framework. In this case, one needs Monte Carlo simulation methods. Recent new Monte Carlo simulation method can be found in Battulga (2024).

4. Conclusion

In this paper, we study Gordon growth model, based on the VAR process, and provide two propositions. The first one deals with two representations of stock price, which arise in generic Gordon growth model and is related to stock forecast. The other one is dedicated to the existence conditions of stock price and its second-order moment for Gordon growth model, based on a simple VAR process. Also, we obtain a result that if one uses the error correction model (cointegrated) for Gordon growth model, the stock price does not exist.

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