Mongolian Mathematical Journal
24 (2023) 24–29

Applications of Fenchel Duality for Vector Optimization Problems

Lkhamsuren Altangerel

Abstract. This paper deals with applications of Fenchel duality for vector optimization problems based on alternative definitions of the conjugate maps and the subgradient for a set-valued map having vector variables (cf. [3]). Some results investigated in Section 4 in [3] allow us to consider applications presented in this paper.

1. Introduction

The conjugate duality in scalar optimization was developed by Rockafellar which provides a unified framework for the duality theory [6]. By introducing concepts of conjugate maps and subgradients based on Pareto efficiency, Sawaragi et al. extended it to the case of multi-objective optimization [5]. The so-called perturbation approach in the conjugate duality in vector optimization with applications to vector variational inequalities was investigated by Altangerel et al. [3] by extending the approach investigated in the scalar case (see [2]). Some special cases of dual problems based on alternative definitions of the conjugate maps and the subgradient for a set-valued map having vector variables were investigated also in [3]. As mentioned in [3], the advantage of considering conjugate maps with vector variables consists in the fact that the corresponding dual problems have a more simple form compared to duals investigated in Section 3 in [3], and they can be easily reduced to the duals for scalar optimization problems. For further research dealing with conjugate duality in vector optimization, we refer to [7–9].

In this paper, we consider some applications of Fenchel duality for vector optimization problems based on alternative definitions of the conjugate maps and the subgradient for a set-valued map having vector variables investigated in [3].

This paper is organized as follows. In Section 2, we recall the definition of the maximum of a set in a finite-dimensional Euclidean space and some of its properties. Moreover, the definitions of conjugate maps, set-valued subgradients with vector variables, and the duality results in vector optimization are given. Section 3 is devoted to the applications of duality results to different problems.

2020 *Mathematics Subject Classification:* 90C29,49N15,49J40.

Key Words and Phrases: Vector optimization problems, Fenchel duality, Optimality conditions.

Received: 2023.03.25; *Accepted:* 2023.05.01; *Published:* 2023.08.31.

2. Fenchel duality for vector optimization

Let $\mathbb{R}^n_+ = \left\{ x = (x_1, ..., x_n)^T \in \mathbb{R}^n |\ x_i \geq 0,\ i = \overline{1,n} \right\}$ be the nonnegative orthant and $Y \subseteq \mathbb{R}^n$ be a given set. For any $\xi, \mu \in \mathbb{R}^n$, we use the following ordering relations:

$$\xi \underset{\mathbb{R}^n_+ \setminus \{0\}}{\leq} \mu \Leftrightarrow \mu - \xi \in \mathbb{R}^n_+ \setminus \{0\};$$

$$\xi \underset{\mathbb{R}^n_+ \setminus \{0\}}{\nleq} \mu \Leftrightarrow \mu - \xi \notin \mathbb{R}^n_+ \setminus \{0\}.$$

The sets of all maximal and minimal points of Y (resp. the maximum and the minimum of Y) are defined by

$$\max_{\mathbb{R}^n_+ \setminus \{0\}} Y := \{x \in Y |\ \nexists x' \in Y \text{ such that } x \underset{\mathbb{R}^n_+ \setminus \{0\}}{\leq} x'\}$$

and

$$\min_{\mathbb{R}^n_+ \setminus \{0\}} Y := \{x \in Y |\ \nexists x' \in Y \text{ such that } x' \underset{\mathbb{R}^n_+ \setminus \{0\}}{\leq} x\},$$

respectively.

Definition 1. *[4, cf. Definition 8.2.2]*

(i) *Let $Y \subseteq \mathbb{R}^n$ be a given set. The set $\min\limits_{\mathbb{R}^n_+ \setminus \{0\}} Y$ is said to be externally stable if*

$$Y \subseteq \min_{\mathbb{R}^n_+ \setminus \{0\}} Y + \mathbb{R}^n_+.$$

(ii) *Similarly, the set $\max\limits_{\mathbb{R}^n_+ \setminus \{0\}} Y$ is said to be externally stable if*

$$Y \subseteq \max_{\mathbb{R}^n_+ \setminus \{0\}} Y - \mathbb{R}^n_+.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a vector-valued function and $G \subseteq \mathbb{R}^n$. Consider the vector optimization problem

$$(VO) \qquad \min_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ f(x) |\ x \in G \right\}.$$

In other words, (VO) is the problem of finding $\bar{x} \in G$ such that

$$f(x) \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleq} f(\bar{x}),\ \forall x \in G.$$

In analogy to the scalar case (see [2]), by introducing some perturbation functions, different dual problems to (VO) have been derived. Hereby the feasible set was given by

$$G = \left\{ x \in X |\ g(x) \underset{\mathbb{R}^m_+}{\leq} 0 \right\},$$

where $X \subseteq \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function.
In this paper we restrict our attention to the Fenchel dual problem to (VO) based on the conjugate map with the vector variable. Before considering this dual, let us recall the definitions of the conjugate maps with vector variables.

Definition 2. *[4, Definition 7.2.3] (the type II Fenchel transform)* *Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map.*

(i) *The set-valued map $h_p^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ defined by*

$$h_p^*(\lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} \Big[(\lambda^T x)_p - h(x)\Big],\ \lambda \in \mathbb{R}^n$$

is called the (type II) conjugate map of h;

(ii) *The conjugate map of h_p^*, h_p^{**} is called the biconjugate map of h, i.e.*

$$h_p^{**}(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda\in\mathbb{R}^n} \Big[(\lambda^T x)_p - h_p^*(\lambda)\Big],\ x \in \mathbb{R}^n;$$

(iii) *$\lambda \in \mathbb{R}^n$ is said to be a subgradient of the set-valued map h at $(\bar{x};\bar{y})$ if $\bar{y} \in h(\bar{x})$ and*

$$\bar{y} - (\lambda^T \bar{x})_p \in \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} [h(x) - (\lambda^T x)_p],$$

where $(\lambda^T x)_p = (\lambda^T x, \ldots, \lambda^T x)^T \in \mathbb{R}^p$.

The set of all subgradients of h at $(x;y)$ is denoted by $\partial h(x;y)$ and is called the *subdifferential of h at $(x;y)$*. If $\partial h(x;y) \neq \varnothing$, $\forall y \in h(x)$, then h is said to be *subdifferentiable at x*.

According to [3], the Fenchel dual to (VO) becomes

$$(D_F^{VO}) \qquad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{t\in\mathbb{R}^n} \Big[- f_p^*(t) + (\min_{x\in G} t^T x)_p \Big].$$

One can notice that to (D_F^{VO}) corresponds the perturbation function defined by

$$\varphi_F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\},\ \varphi_F(x,v) = \begin{cases} f(x+v), & x \in G, \\ +\infty, & \text{otherwise.} \end{cases}$$

The corresponding value function can be written as

$$\begin{aligned} \psi_F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p,\ \psi_F(v) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\varphi_F(x,v) \mid x \in \mathbb{R}^n\Big\} \\ &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{f(x+v) \mid x \in G\Big\}. \end{aligned}$$

More details about the conjugate duality based on Pareto efficiency and the perturbation approach on this theory can be found in [1], and [3].
The problem (VO) is said to be stable with respect to the perturbation function φ_F, if the value function ψ_F is subdifferentiable at 0.

Proposition 1. *[3, Proposition 4.4]*

(i) *The problem (VO) is stable with respect to φ_F if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{t} \in \mathbb{R}^n$ to the dual problem (D_F^{VO}) such that*

$$f(\bar{x}) \in -f_p^*(\bar{t}) + (\min_{x\in G} \bar{t}^T x)_p \tag{2.1}$$

and $\bar{t}^T \bar{x} = \min_{x\in G} \bar{t}^T x$.

(ii) *Conversely, if* $\bar{x} \in G$ *and* $\bar{t} \in \mathbb{R}^n$ *satisfy the above conditions, then* $\bar{x}$ *and* $\bar{t}$ *are solutions to* (VO) *and* (D_F^{VO})*, respectively.*

3. Applications

3.1. The vector optimization problem with linear objective function

Let $C \in \mathbb{R}^{p\times n}$ and $G \subseteq \mathbb{R}^n$. Consider the vector optimization problem

$$(P_C) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{Cx|\ x \in G\Big\}.$$

Before applying Proposition 1 to this case, let us give the following auxiliary result.

LEMMA 1. *[3, Lemma 5.1] Let* $M \in \mathbb{R}^{p\times n}$*. Then*

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \{My|\ y \in \mathbb{R}^n\} = \begin{cases} \{My|\ y \in \mathbb{R}^n\}, & \text{if } \exists \mu \in int\ \mathbb{R}^p_+ \text{ such that } \mu^T M = 0^T, \\ \varnothing, & \text{otherwise.} \end{cases}.$$

REMARK 1. *Let* $t \in \mathbb{R}^n$ *be fixed. Taking* $f(x) = Cx$ *and by Lemma 1, we obtain that*

$$\begin{aligned} f_p^*(t) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \{(t^T v)_p - f(v)|\ v \in \mathbb{R}^n\} = \max_{\mathbb{R}^p_+\setminus\{0\}} \{(t^T v)_p - Cv|\ v \in \mathbb{R}^n\} \\ &= -\min_{\mathbb{R}^p_+\setminus\{0\}} \{(C - B)v|\ v \in \mathbb{R}^n\} = \{(B - C)v|\ v \in \mathbb{R}^n\}, \end{aligned}$$

where $B := [t, ..., t]^T \in \mathbb{R}^{p\times n}$ *and* $t \in N := \{t \in \mathbb{R}^n|\ \exists \mu \in int\ \mathbb{R}^p_+ \text{ such that } \mu^T(C - B) = 0\}$.

In view of Remark 1, the Fenchel dual to (P_C) turns out to be

$$(D_F^C) \qquad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{t\in N} \Big[\{(C - B)v|\ v \in \mathbb{R}^n\} + (\min_{x\in G} t^T x)_p\Big].$$

PROPOSITION 2. *Let the function* $f : \mathbb{R}^n \to \mathbb{R}^p$ *be defined by* $f(x) = Cx$*, where* $C \in \mathbb{R}^{p\times n}$.

(i) *The problem* (P_C) *is stable with respect to* φ_F *if and only if for each solution* $\bar{x}$ *to* (P_C)*, there exists a solution* $\bar{t} \in N$ *to the dual problem* (D_F^C) *such that*

$$C\bar{x} \in \{(C - \bar{B})v|\ v \in \mathbb{R}^n\} + (\min_{x\in G} \bar{t}^T x)_p \tag{3.1}$$

and

$$\bar{t}^T \bar{x} = \min_{x\in G} \bar{t}^T x. \tag{3.2}$$

(ii) *Conversely, if* $\bar{x} \in G$ *and* $\bar{t} \in N$ *satisfy* (3.1)–(3.2)*, then* $\bar{x}$ *and* $\bar{t}$ *are solutions to* (P_C) *and* (D_F^C)*, respectively.*

Let us now give a stability criterion for (P_C).

Proposition 3. *Let the set* $\min\limits_{\mathbb{R}^p_+\setminus\{0\}}\{Cx|\ x\in G\}$ *be externally stable. Then the problem* (P_C) *is stable with respect to* φ_F.

3.2. The vector variational inequality

Let $F:\mathbb{R}^n\to\mathbb{R}^{n\times p}$ be a matrix-valued function and $K\subseteq\mathbb{R}^n$. The vector variational inequality problem consists in finding $x\in K$ such that

$$(VVI)\qquad F(x)^T(y-x)\ \nleq_{\mathbb{R}^p_+\setminus\{0\}}\ 0,\ \forall y\in K.$$

One can notice that $x\in K$ is a solution to the problem (VVI) if and only if 0 is a minimal point of the set $\Big\{F(x)^T(y-x)|\ y\in K\Big\}$. In other words, x is a solution to the following vector optimization problem

$$(P^{VVI};x)\qquad \min_{\mathbb{R}^p_+\setminus\{0\}}\Big\{F(x)^T(y-x)|\ y\in K\Big\}.$$

For a fixed $x\in K$ we consider the problem $(P^{VVI};x)$. Since the function $y\mapsto F(x)^T(y-x)$ is linear, one can apply duality results to this problem in analogy to Subsection 3.1. Let $D:=[t,...,t]\in\mathbb{R}^{n\times p}$ and for a fixed $x\in K$ the set $N(x)$ be defined by

$$N(x):=\{t\in\mathbb{R}^n|\ \exists\mu\in\text{int }\mathbb{R}^p_+\ \text{ such that }\ (F(x)-D)\mu=0\}.$$

Taking $y\mapsto F(x)^T(y-x)$ as the objective function, by Lemma 1, one has

$$(D_F^{VVI};x)\qquad \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{t\in N(x)}\Big\{-F(x)^Tx+\{(F(x)-D)^Ty|\ y\in\mathbb{R}^n\}+(\min_{y\in K}t^Ty)_p\Big\}.$$

Proposition 4. *Let for any* $x\in K$ *the function* $f:\mathbb{R}^n\to\mathbb{R}^p$ *be defined by* $f(y)=F(x)^T(y-x)$, *where* $F:\mathbb{R}^n\to\mathbb{R}^{n\times p}$ *is a matrix-valued function.*

(i) *For a solution* $\bar{x}\in K$ *to* (VVI) *the problem* $(P^{VVI};\bar{x})$ *is stable with respect to* φ_F *if and only if there exists a solution* $\bar{t}\in N(\bar{x})$ *to the dual problem* $(D_F^{VVI};\bar{x})$ *such that*

$$0\in -F(\bar{x})^T\bar{x}+\{(F(\bar{x})-\overline{D})^Ty|\ y\in\mathbb{R}^n\}+(\min_{y\in K}\bar{t}^Ty)_p \tag{3.3}$$

and

$$\bar{t}^T\bar{x}=\min_{x\in K}\bar{t}^Ty, \tag{3.4}$$

where $\overline{D}:=[\bar{t},...,\bar{t}]\in\mathbb{R}^{n\times p}$.

(ii) *Conversely, if* $\bar{x}\in K$ *and* $\bar{t}\in\mathbb{R}^n$ *satisfy* (3.3)–(3.4), *then* $\bar{x}$ *and* $\bar{t}$ *are solutions to* (VVI) *and* $(D_F^{VVI};\bar{x})$, *respectively.*

References

1. Altangerel, L., *A duality approach to gap functions for variational inequalities and equilibrium problems,* Ph.D. Dissertation, Faculty of Mathematics, Chemnitz University of Technology, 2006.
2. Altangerel, L.; Boţ, R.I.; Wanka, G., *On the construction of gap functions for variational inequalities via conjugate duality,* Asia-Pacific Journal of Operational Research 24 (03), 353-371, 2007

3. Altangerel, L.; Boţ, R.I.; Wanka, G., *Conjugate Duality in Vector Optimization and some Applications to the Vector Variational Inequality,* Journal of Mathematical Analysis and Applications 329 (2), 1010–1035, 2007.
4. Goh, C.J.; Yang, X.Q., *Duality in optimization and variational inequalities,* London: Taylor and Francis, 2002.
5. Sawaragi, Y.; Nakayama, H.; Tanino, T., *Theory of multiobjective optimization,* Mathematics in Science and Engineering, Vol. 176. Orlando etc.: Academic Press, 1985.
6. Rockafellar, R. *Convex analysis,* Princeton Mathematical Series Princeton University Press, 1970.
7. Tanino, T.; Sawaragi, Y., *Conjugate maps and duality in multiobjective optimization,* Journal of Optimization Theory and Applications 31, 473-499, 1980.
8. Tanino, T., *On supremum of a set in a multidimensional space,* Journal of Mathematical Analysis and Applications 130, No.2, 386-397, 1988.
9. Tanino, T., *Conjugate duality in vector optimization,* Journal of Mathematical Analysis and Applications 167, No.1, 84-97, 1992.

Lkhamsuren Altangerel
German-Mongolian Institute for Resources and Technology
Ulaanbaatar, Mongolia

altangerel@gmit.edu.mn