Mongolian Mathematical Journal
24 (2023) 17–23

Optimal Control Sphere Packing Problem

Rentsen Enkhbat

Abstract. The sphere packing problem which is to pack non-overlapping spheres with the maximum volume into a convex set. The problem belongs to a class of global optimization. Convex maximization formulation of the problem is given in [1, 12]. In this paper, we formulate a new optimal control problem based on the sphere packing problem which is a nonconvex optimal control problem with phase and control constraints. A discrete version of the new optimal control problem for sphere packing problem has been discussed. We examine also Malfatti's problem [17] from a view point of optimal control theory.

1. Introduction

The sphere packing problem is one of the oldest problems in mathematics and finds many applications in science and technology [1-8].
Recently, new applications of sphere packing problem have been appeared in mining industry [13], finance and economics [16].
In general, the sphere packing problems are reduced to difficult nonconvex optimization problems which cannot be handled effectively by analytical approaches. In [12], it has been shown that a general sphere packing problem belongs to a class of convex maximization problem over a nonconvex set. Moreover, in [12] it has been shown that the classical circle packing problem as well as Malfatti's problem are particular cases of the sphere packing problem. The complexity of the sphere packing problems increases rapidly as a number of spheres increases. Thus, only heuristic type methods are available for high dimensional cases ($N > 50$). In [14], [15] the sphere packing problem was examined from a view point of game theory and multi-criteria optimization.
It seems that so far there is no work on optimal control formulataion of sphere packing problem. Aim of this paper is to fulfill this gap.

2. Sphere Packing Problem

Let $B(x^0, r)$ be a ball with a center $x^0 \in \mathbb{R}^n$ and radius $r \in \mathbb{R}$.

$$B(x^0, r) = \{x \in \mathbb{R}^n | \|x - x^0\| \le r\}, \tag{2.1}$$

here $\|\cdot\|$ is Euclidean norm.

2020 *Mathematics Subject Classification:* 49M25, 49N15.

Key Words and Phrases: Sphere packing problem, optimal control, phase and control constraints, Malfatti's problem.

Received: 2022.11.15; *Accepted:* 2023.05.30; *Published:* 2023.08.31.

The n-dimensional volume of the Euclidean ball $B(x^0, r)$ is [18, 19]:

$$V(B) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} r^n, \tag{2.2}$$

where Γ is Leonhard Euler's gamma function. Let D be a polyhedral set given by the following linear inequalities.

$$D = \{x \in \mathbb{R}^n | \langle a^i, x\rangle \le b_i, i = \overline{1,m}\}, a^i \in \mathbb{R}^n, b_i \in \mathbb{R},$$

where $\langle , \rangle$ denotes the scalar product of two vectors in $\mathbb{R}^n$.

Assume that D is compact and $intD \neq \emptyset$. Clearly, D is a convex set in $\mathbb{R}^n$.

Denote by $x^1, x^2, \dots x^k$ centers of the spheres inscribed in D. Let $u_1, u_2, ..., u_k$ be their corresponding radii.

Now we consider a problem of maximizing a total volume of k non-overlapping spheres inscribed in $D \subset \mathbb{R}^n$. This problem in the literature is often called sphere packing problem for different spheres.

Now we formulate the following optimization problem :

$$\max_{(x,u)} V = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \sum_{i=1}^{k} u_i^n \tag{2.3}$$

subject to:

$$\langle a^i, x^j\rangle + u_j\|a^i\| \le b_i, i = \overline{1,m}, j = \overline{1,k}, \tag{2.4}$$

$$\|x^i - x^j\|^2 \ge (u_i + u_j)^2, i, j = \overline{1,k},\ i < j, \tag{2.5}$$

$$u_1 \ge 0, u_2 \ge 0, ..., u_k \ge 0. \tag{2.6}$$

The function V, which is convex as a sum of convex functions u_i^n defined on the positive orthant of $\mathbb{R}$, denotes the total volume of all spheres inscribed in D. Conditions (2.4) describe that all spheres are in D while conditions (2.5) secure non-overlapping conditions for spheres.

If we set $n = 2, m = 3$ and $k = 3$ in problem (2.3)-(2.6), then the problem becomes Malfatti's problem which was first formulated in 1803 [17]. Indeed, the problem has the form:

$$\max_{(u,r)} V = \pi \sum_{i=1}^{3} r_i^2 \tag{2.7}$$

subject to:

$$\langle a^i, u^j\rangle + r_j\|a^i\| \le b_i, i, j = 1, 2, 3, \tag{2.8}$$

$$\|u^i - u^j\|^2 \ge (r_i + r_j)^2, i, j = 1, 2, 3, i \neq j, \tag{2.9}$$

$$r_1 \ge 0, r_2 \ge 0, r_3 \ge 0. \tag{2.10}$$

The original Malfatti's and its extended four circle problem as well three dimensional Malfatti's problem were solved numerically in [9–11].

3. Optimal Control Sphere Packing

Let us consider a problem of maximizing a total volume of k non-overlapping spheres inscribed in $D \subset \mathbb{R}^n$ centers of which move along trajectories given by differential equations. These trajectories are:

$$\begin{cases} \dot{x}^j = f^j(x^j, t), \quad x^j \in \mathbb{R}^n, \\ x^j(0) = x_0^j, \quad j = 1, 2, \ldots, k, \end{cases} \tag{3.1}$$

$$x^j = (x_1^j, x_2^j, \ldots, x_n^j), \quad j = 1, 2, \ldots, k,$$

where $f^j : \mathbb{R}^{n+1} \to \mathbb{R}^k$ are vector-valued differentiable mappings, $f^j = (f_1^j, f_2^j, \ldots, f_k^j)$.

Denote by $u_1(t), \ldots, u_k(t)$ radii of spheres at moment t.

Assume that $u_j : R \to R, j = 1, 2, \ldots, k$ are continuously differentiable functions. Let $B(x^0(t), u^0(t))$ be a ball with a center $x^0(t) \in \mathbb{R}^n$ and radius $u^0(t)$ at moment t.

$$B(x^0, u^0) = \{x \in \mathbb{R}^n \mid \|x - x^0\| \le u^0\}.$$

Then condition $B(x^0, u^0) \subset D$ becomes [12]:

$$\langle a^q, x^0(t)\rangle + u^0(t)\|a^q\| \le b_q, q = 1, 2, \ldots, m. \tag{3.2}$$

Non-overlapping conditions for spheres $B(x^i, u_i), i = 1, 2, \ldots, k$ are

$$\|x^i - x^j\|^2 \ge (u_i + u_j)^2, \quad i, j = 1, 2, \ldots, k, \quad i < j. \tag{3.3}$$

The total volume of spheres inscribed in D at the moment t is

$$V(t) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \sum_{j=1}^{k} u_j^n(t),$$

$$u(t) = (u_1(t), \ldots, u_k(t)) \in \mathbb{R}^n, \quad t \in (-\infty, +\infty).$$

Then under the above assumptions, the volume maximization problem is formulated as follows:

$$\max_{(u,T,t_0)} J = \max_{(u,T,t_0)} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \sum_{j=1}^{k} u_j^n(T) \tag{3.4}$$

subject to constraints:

$$\langle a^q, x^j(t)\rangle + u_j(t)\|a^q\| \le b_q, \tag{3.5}$$

$$t_0 \le t \le T, \quad q = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, k.$$

$$\begin{cases} \dot{x}^j = f^j(x^j, t), \\ x^j(t_0) = x_0^j, \quad j = 1, 2, \ldots, k, \quad t_0 \le t \le T. \end{cases}$$

$$\|x^i(t) - x^j(t)\|^2 \ge (u_i(t) + u_j(t))^2, \quad i, j = 1, 2, \ldots, k, \quad i < j, \tag{3.6}$$

$$u_1(t) \ge 0, u_2(t) \ge 0, \ldots, u_k(t) \ge 0.$$

$$\begin{cases} \langle a^q, x^j(T)\rangle \le b_q \\ \langle a^q, x_0^j(t_0)\rangle \le b_q, \qquad q = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, k. \end{cases} \tag{3.7}$$

Problem (3.4)-(3.7) can be written in the form:

$$\max_{(u,T,t_0)} J = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \sum_{j=1}^{k} u_j^n(T) \tag{3.8}$$

$$\langle a^q, x_0^j + \int_{t_0}^{t} f^j(x^j,\tau)\mathrm{d}\tau\rangle + u_j(t)\|a^q\| \le b_q, \tag{3.9}$$

$$t_0 \le t \le T, \quad q = 1,2,\dots,m, \quad j = 1,2,\dots,k,$$

$$\left\| \int_{t_0}^{t} \left[f^j(x^j,\tau) - f^i(x^i,\tau) \right] \mathrm{d}\tau + (x_0^j - x_0^i) \right\|^2 \ge (u_i(t) + u_j(t))^2, \tag{3.10}$$

$$i,j = 1,2,\dots,k, \quad i < j.$$

$$\begin{cases} \left\langle a^q, x_0^j + \int_{t_0}^{T} f^j(x^j,\tau)\mathrm{d}\tau \right\rangle \le b_q \\ \langle a^q, x_0^j \rangle \le b_q, \qquad q = 1,2,\dots,m, \quad j = 1,2,\dots,k, \end{cases} \tag{3.11}$$

where

$$\int_{t_0}^{T} f^j(x^j,\tau)\mathrm{d}\tau = (\int_{t_0}^{T} f_1^j(x^j,\tau)\mathrm{d}\tau, \dots, \int_{t_0}^{T} f_k^j(x^j,\tau)\mathrm{d}\tau).$$

Now we can write the discrete optimal control sphere packing problem

$$\max\ J = \max_{(u,t_p)} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \sum_{j=1}^{k} u_j^n(t_p), \quad p \in \{0,1,\dots,N\}$$

$$\langle a^q, x^j(t_{p+1})\rangle + u_j(t_{p+1})\|a^q\| \le b_q,$$

$$q = 1,2,\dots,m, \quad j = 1,2,\dots,k, \quad p = 0,1,\dots,N-1.$$

$$\begin{cases} x^j(t_{p+1}) = x^j(t_p) + f^j(x^j(t_p)), \quad j = 1,2,\dots,k, \\ x^j(t_0) = x_0^j, \quad p = 0,1,\dots,N-1. \end{cases}$$

$$\left\| x^i(t_p) - x^j(t_p) \right\|^2 \ge \left(u_i(t_p) + u_j(t_p) \right)^2,$$
$$p = 0,1,\dots,N-1, \quad i,j = 1,2,\dots,k, \quad i < j.$$

$$\begin{cases} \langle a^q, x^j(N)\rangle \le b_q, \quad q = 1,2,\dots,m, \\ \langle a^q, x_0^j\rangle \le b_q, \quad j = 1,2,\dots,k. \end{cases}$$

Denote by $u_j(t_p) = u_{jp}$, $x^j(t_{p+1}) = x_{p+1}^j$, $u_j(t_{p+1}) = u_{jp+1}$, $x^j(t_p) = x_p^j$, $f^j(x^j(t_p)) = f^j(x_p^j)$, $u_i(t_p) = u_{ip}$, $x^i(N) = x_{iN}, i,j = 1,2,,\dots,k;$ $i < j$. Then the problem is equivalent to

$$\max_{(u,t_0,N)} J = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \sum_{j=1}^{k} u_{jN}^n,$$

$$\left\langle a^q, x_0^j + \int_{t_0}^{t_p} f^j(x^j,\tau)\mathrm{d}\tau \right\rangle + u_{jp}\|a^q\| \le b_q,$$
$$p = 0,1,\dots,N-1, \quad j = 1,2,\dots,k, \quad q = 1,2,\dots,m.$$

$$\begin{cases} x^j_{p+1} - x^j_p - f^j(x^j_p) = 0, & j = 1, 2, \ldots, k, \\ x^j(t_0) = x^j_0 = x_{j0}, & p = 0, 1, \ldots, N-1. \end{cases}$$

$$\|x^i_p - x^j_p\|^2 \geq (u_{ip} + u_{jp})^2, \quad i, j = 1, 2, \ldots, k, \quad i < j,$$

$$\begin{cases} \langle a^q, x^j_N \rangle - b_q \leq 0, & q = 1, 2, \ldots, m, \\ \langle a^i, x^j_0 \rangle \leq b_q, & j = 1, 2, \ldots, k. \end{cases}$$

Since Malfatti's problem is a particular case of problem (3)-(7) for $k = 3, m = 3, n = 2$, then Malfatti's optimal control problem can be formulated as follows.

$$\max_{(u,T,t_0)} J = \pi(u_1^2(T) + u_2^2(T) + u_3^2(T))$$

subject to constraints:

$$\begin{gathered} \langle a^1, x^1(t) \rangle + u_1(t)\|a^1\| \leq b_1, \\ \langle a^2, x^2(t) \rangle + u_2(t)\|a^2\| \leq b_2, \\ \langle a^3, x^3(t) \rangle + u_3(t)\|a^3\| \leq b_3, \\ t_0 \leq t \leq T < +\infty, \\ \begin{cases} \frac{\mathrm{d}x^1}{\mathrm{d}t} = f^1(x^1, t), \\ x^1(t_0) = x^1_0 \end{cases} \\ \begin{cases} \frac{\mathrm{d}x^2}{\mathrm{d}t} = f^2(x^2, t), \\ x^2(t_0) = x^2_0 \end{cases} \\ \begin{cases} \frac{\mathrm{d}x^3}{\mathrm{d}t} = f^3(x^3, t), \\ x^3(t_0) = x^3_0 \end{cases} \\ \|x^1(t) - x^2(t)\|^2 \geq (u_1(t) + u_2(t))^2 \\ \|x^1(t) - x^3(t)\|^2 \geq (u_1(t) + u_3(t))^2 \\ \|x^2(t) - x^3(t)\|^2 \geq (u_3(t) + u_2(t))^2 \end{gathered}$$

$$\begin{gathered} \langle a^1, x^1(T) \rangle \leq b_1 \\ \langle a^2, x^2(T) \rangle \leq b_2 \\ \langle a^3, x^3(T) \rangle \leq b_3 \\ \langle a^1, x^1_0(T) \rangle \leq b_1 \\ \langle a^2, x^2_0(T) \rangle \leq b_2 \\ \langle a^3, x^3_0(T) \rangle \leq b_3 \\ u_1(t) \geq 0, u_2(t) \geq 0, u_3(t) \geq 0, \end{gathered}$$

where

$$\begin{gathered}
f^1 = (f_1^1(x^1, t), f_2^1(x^1, t)), \\
f^2 = (f_1^2(x^2, t), f_2^2(x^2, t)), \\
f^3 = (f_1^3(x^3, t), f_2^3(x^3, t)), \\
x^1(t) = (x_1^1(t), x_2^1(t)), \\
x^2(t) = (x_1^2(t), x_2^2(t)), \\
x^3(t) = (x_1^3(t), x_2^3(t)), \\
-\infty < t_0 \le t \le T < +\infty.
\end{gathered}$$

Conclusion

For the first time, the general sphere packing problem which is to pack non-overlapping spheres into a convex set with the maximum volume has been examined from a view point of optimal control theory. The new problem is a difficult nonconvex optimal control problem with mixed constraints with respect to trajectory and control variables. We also formulate a discrete version of the problem. Computational methods and algorithms for solving the new optimal control sphere packing problem will be discussed in a next paper.

References

1. Enkhbat, R. 2022. "Recent Advances in Sphere Packing Problems." in Y.Kochetov, A.Eremeev, O.Khamisov, A.Rettieva (Eds.): Mathematical Optimization Theory and Operations Research, Springer, 34-35.
2. Nguyen, V. H., and Strodiot, J. J. 1992. "Computing a Global Optimal Solution to a Design Centering Problem." Mathematical Programming 53: 111-123.
3. Vidigal, L. M., and Director, S. W. 1982. "A Design Centering Algorithm for Nonconvex Regions of Acceptability." IEEE Transactions on Computer-Aided-Design of Integrated Circuits and Systems, CAD-1: 13-24.
4. Hifi, M., and M'Hallah, R. 2004. "Approximate algorithms for constrained circular cutting problems." Computers and Operations Research 31: 675-694.
5. Fraser, H. J., and George, J. A. 1994. "Integrated container loading software for pulp and paper industry." European Journal of Operational Research 77: 466-474.
6. Dowsland, K. A. 1991. "Optimising the palletisation of cylinders in cases." OR Spectrum 13: 204-212.
7. Erkut, E. 1990. "The discrete p-dispersion problem." European Journal of Operational Research 46: 48-60.
8. Drezner, Z., and Erkut, E. 1995. "Solving the continuous p-dispersion problem using non-linear programming." Journal of the Operational Research Society 46: 516-520.
9. Enkhbat, R. 2016. "Global Optimization Approach to Malfatti's Problem." Journal of Global Optimization, 65: 33-39.
10. Enkhbat, R., Barkova, M., and Strekalovsky, A. 2016. "Solving Malfatti's High Dimensional Problem by Global Optimization." Numerical Algebra, Control and Optimization 6(2): 153-160.
11. Enkhbat, R., and Barkova, M. 2016. "Global Search Method for Solving Malfatti's four-Circle Problem." The Bulletin of Irkutsk State University, Series "Mathematics" 15: 38-49.
12. Enkhbat, R. 2020. "Convex Maximization Formulation of General Sphere Packing Problem" The Bulletin of Irkutsk State University, Series "Mathematics" 31: 142-149.
13. Enkhbat, R., Tungalag, N., Enkhbayar, J., Battogtokh, O., and Enkhtuvshin, L. 2021. "Application of Survival Theory in Mining Industry." Numerical Algebra, Control and Optimization 11(3): 443-448.
14. Enkhbat, R., and Battur, G. 2021. "Multi-objective Optimization Approach to Malfatti's Problem." Contributions to Game Theory and Management 14, Saint Petersburg State University, 82-90.

15. Enkhbat, R., and Battur, G. 2021. " Generalized Nash Equilibrium Based on Malfatti's Problem." Numerical Algebra, Control and Optimization 11(2): 209-220.
16. Badam, U., Enkhbat, R., and Batchimeg, Ts. 2021. "Application of Survival Theory in a Taxation." Journal of Industrial and Management Optimization 15(5): 2573-2578.
17. Malfatti, G. 1803. "Memoria sopra una problema stereotomico." Memoria di Matematica e di Fisica della Societa italiana della Scienze, 10(1): 235-244.
18. Paris, R. B., and Kaminski, D.2001. "Asymptotics and Mellin-Barnes Integrals." Cambridge University Press.
19. Equation 5.19.4, NIST Digital Library of Mathematical Functions,http://dlmf.nist.gov/5.19E4, 2013.

Rentsen Enkhbat
Institute of Mathematics and Digital Technology
Mongolian Academy of Sciences
Ulaanbaatar 13330, Mongolia

renkhbat46@yahoo.com