Mongolian Mathematical Journal
24 (2023) 1–16

Rainbow Options with MS–VAR process

Battulga Gankhuu

Abstract. This paper presents pricing and hedging methods for rainbow options and lookback options under Markov-Switching Vector Autoregressive (MS–VAR) process. Here we assumed that a regime–switching process is generated by a homogeneous Markov process. An advantage of our model is it depends on economic variables and simple as compared with previous existing papers.

1. Introduction

The first option pricing formula dates back to classic papers of [6] and [21]. They implicitly introduced a risk-neutral valuation method to arbitrage pricing. But it was not fully developed and appreciated until the works of [17] and [18]. The basic idea of the risk-neutral valuation method is that discounted price process of an underlying asset is a martingale under some risk-neutral probability measure. The option price is equal to an expected value, with respect to the risk-neutral probability measure, of discounted option payoff. In this paper, to price rainbow options and lookback options, we use the risk-neutral valuation method in the presence of economic variables.

Sudden and dramatic changes in the financial market and economy are caused by events such as wars, market panics, or significant changes in government policies. To model those events, some authors used regime–switching models. The regime–switching model was introduced by seminal works of [14, 15] (see also books of [16] and [20]) and the model is hidden Markov model with dependencies, see [29]. Markov regime–switching models have been introduced before Hamilton (1989), see, for example, [13], [23], and [28]. The regime–switching model assumes that a discrete unobservable Markov process generates switches among a finite set of regimes randomly and that each regime is defined by a particular parameter set. The model is good fit for some financial data and has become popular in financial modeling including equity options, bond prices, and others.

Economic variables play important role in any economic model. In some existing option pricing models, the underlying asset price is governed by some stochastic process and it has not taken into account economic variables such as GDP, inflation, unemployment rate, and so on. For example, the classical Black-Scholes option pricing model uses a geometric Brownian motion to capture underlying asset prices. However, the underlying asset price modeled by geometric Brownian motion is not a realistic assumption when it comes to option pricing. In reality, for the Black-Scholes model, the price process of the asset should depend on some economic variables.

2020 *Mathematics Subject Classification:* 91B84, 91G20, 91G30.

Key Words and Phrases: Rainbow and lookback options, Bayesian MS-VAR process, locally risk-minimizing strategy.

Received: 2022.10.22; *Accepted:* 2022.11.28; *Published:* 2023.08.31.

Classic Vector Autoregressive (VAR) process was proposed by [25] who criticize large–scale macroeconometric models, which are designed to model inter–dependencies of economic variables. Besides [25], there are some other important works on multiple time series modeling, see, e.g., [27], where a class of vector autoregressive moving average models was studied. For the VAR process, a variable in the process is modeled by its past values and past values of other variables in the process. After the work of [25], VARs have been used for macroeconomic forecasting and policy analysis. However, if the number of variables in the system increases or the time lag is chosen high, then too many parameters need to be estimated. This will reduce the degrees of freedom of the model and entails a risk of over-parametrization.

Therefore, to reduce the number of parameters in a high-dimensional VAR process, [8] introduced probability distributions for coefficients that are centered at the desired restrictions but that have a small and nonzero variance. Those probability distributions are known as Minnesota prior in Bayesian VAR (BVAR) literature which is widely used in practice. Due to over–parametrization, the generally accepted result is that forecast of the BVAR model is better than the VAR model estimated by the frequentist technique. Research works have shown that BVAR is an appropriate tool for modeling large data sets, for example, see [2].

In this paper, to partially fill the gaps mentioned above, we introduced a Markov–Switching VAR (MS–VAR) model to value and hedge the options. Our model offers the following advantages: (i) it tries to mitigate valuation complexity of previous rainbow option models with regime–switching (ii) it considers economic variables thus the model will be more consistent with future economic uncertainty (iii) it introduces regime–switching so that the model takes into account sudden and dramatic changes in the economy and financial market (iv) it adopts a Bayesian procedure to deal with over–parametrization. Novelty of the paper is that we introduced MS–VAR process which is widely used to model economic variables to rainbow options and lookback options.

The rest of the paper is structured as follows. In Section 2, we will consider some results, which include a Theorem used to price and hedge the rainbow options and lookback options and a log–normal system of economic and financial variables in [4]. The author obtained pricing formulas for some frequently used options under MS–VAR process. Section 3 is devoted to pricing the rainbow options and lookback options. Section 4 provides hedging formulas which are based on the locally risk–minimizing strategy for the options. Finally, Section 5 concludes the study.

2. Review

In this section, we will consider some results in [4]. Let $(\Omega, \mathcal{H}_T, \mathbb{P})$ be a complete probability space, where $\mathbb{P}$ is a given physical or realworld probability measure and $\mathcal{H}_T$ will be defined below. To introduce a regime–switching process, we assume that $\{s_t\}_{t=1}^T$ is a homogeneous Markov chain with N state and $\mathsf{P} := \{p_{ij}\}_{i,j=1}^N$ is a random transition probability matrix. We consider a Markov–Switching Vector Autoregressive (MS–VAR(p)) process of p order, which is given by the following equation

$$y_t = A_0(s_t)\psi_t + A_1(s_t)y_{t-1} + \cdots + A_p(s_t)y_{t-p} + \xi_t,\ t = 1, \dots, T, \tag{2.1}$$

where $y_t = (y_{1,t}, \dots, y_{n,t})^T$ is an $(n \times 1)$ random vector, $\psi_t = (1, \psi_{2,t}, \dots, \psi_{k,t})^T$ is a $(k \times 1)$ random vector of exogenous variables, $\xi_t = (\xi_{1,t}, \dots, \xi_{n,t})^T$ is an $(n \times 1)$ Gaussian white noise process with zero mean vector and positive definite random covariance matrix $\Sigma(s_t)$, $A_0(s_t)$ is an $(n \times k)$ is an random coefficient matrix at regime s_t that corresponds to the vector of exogenous variables, for $i = 1, \dots, p$, $A_i(s_t)$ are random $(n \times n)$ coefficient matrices at regime s_t that correspond to the vectors $y_{t-1}, \dots, y_{t-p}$. In this paper, we focused homogeneous MS–VAR process and for heteroscedastic MS–VAR process, we refer to [4]. Equation (2.1) can be compactly written by

$$y_t = \Pi(s_t)\mathsf{Y}_{t-1} + \xi_t, \ t = 1, \dots, T, \tag{2.2}$$

where $\Pi(s_t) := [A_0(s_t) : A_1(s_t) : \dots : A_p(s_t)]$ is random a coefficient matrix at regime s_t which consist of all the coefficient matrices and $\mathsf{Y}_{t-1} := (\psi_t, y_{t-1}^T, \dots, y_{t-p}^T)^T$ is a vector which consist of exogenous variables ψ_t and last p lagged values of the process y_t. In the paper, this form of the MS–VAR process y_t will play a major role than the form which is given by equation (2.1).

Let us introduce stacked vectors and matrices: $y := (y_1', \dots, y_T')'$, $s := (s_1, \dots, s_T)'$, $\Pi := [\Pi(s_1) : \dots : \Pi(s_T)]$, and $\Gamma := [\Sigma(s_1) \cdots : \Sigma(s_T)]$. We also assume that the white noise process $\{\xi_t\}_{t=1}^T$ is independent of the random coefficient matrices Π and Γ, random transition matrix P and regime–switching process $\{s_t\}_{t=1}^T$ conditional on initial information $\mathcal{F}_0 := \sigma(\mathcal{Y}_0, \psi_1, \dots, \psi_T)$. Here for a generic random vector X, $\sigma(X)$ denotes a σ–field generated by X random vector, $\psi_1, \dots, \psi_T$ are values of exogenous variables and they are known at time zero, and for the MS–VAR(p) process, $\mathcal{Y}_0 := (y_{1-p}', \dots, y_0')'$ is an initial value vector of the process y_t and for the Bayesian MS–VAR(p) process, $\mathcal{Y}_0 := (y_{1-p-T_*}', \dots, y_0', \psi_{1-T_*}', \dots, \psi_0')'$ is the data, covering the period T_* before time 1 for the posterior distribution. We further suppose that the transition probability matrix P is independent of the random coefficient matrices Π and Γ given initial information $\mathcal{F}_0$ and regime–switching process s.

To ease of notations, for a generic matrix $O = [O_1 : \dots : O_T]$, we denote its first t and last $T - t$ block matrices by $\bar{O}_t$ and $\bar{O}_t^c$, respectively, that is, $\bar{O}_t := [O_1 : \dots : O_t]$ and $\bar{O}_t^c := [O_{t+1} : \dots : O_T]$. This notation also holds for vectors. We define σ–fields: for $t = 0, \dots, T$, $\mathcal{F}_t := \mathcal{F}_0 \vee \sigma(\bar{y}_t)$, $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\Pi) \vee \sigma(\Gamma) \vee \sigma(\mathsf{P}) \vee \sigma(\bar{s}_t)$ and $\mathcal{H}_t := \mathcal{F}_t \vee \sigma(\Pi) \vee \sigma(\Gamma) \vee \sigma(\mathsf{P}) \vee \sigma(s)$ and for $t = 1, \dots, T$, $\mathcal{I}_{t-1} = \mathcal{F}_{t-1} \vee \sigma(\bar{\Pi}_t) \vee \sigma(\bar{\Gamma}_t) \vee \sigma(\mathsf{P}) \vee \sigma(\bar{s}_t)$, where for generic sigma fields $\mathcal{O}_1, \dots, \mathcal{O}_k$, $\vee_{i=1}^k \mathcal{O}_i$ is the minimal σ–field containing the σ–fields $\mathcal{O}_i$, $i = 1, \dots, k$. Observe that $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$ and $\mathcal{I}_{t-1} \subset \mathcal{G}_t$ for $t = 1, \dots, T$. The σ-fields play major roles in the paper. For the first–order Markov chain, a conditional probability that the regime at time $t+1$, s_{t+1} equals some particular value conditional on the past regimes, $s_t, s_{t-1}, \dots, s_1$ depends only through the most recent regime at time t, s_t, that is,

$$p_{s_t s_{t+1}} := \mathbb{P}(s_{t+1} = s_{t+1} | s_t = s_t, \mathsf{P}, \mathcal{F}_0) = \mathbb{P}\big(s_{t+1} = s_{t+1} | \bar{s}_t = \bar{s}_t, \mathsf{P}, \mathcal{F}_0\big) \tag{2.3}$$

for $t = 0, \dots, T-1$, where $p_{s_1} := p_{s_0 s_1} = \mathbb{P}(s_1 = s_1 | \mathsf{P})$ is an initial probability.

2.1. Risk Neutral Measure

We assume that for $t = 1, \dots, T$, $\mathcal{I}_{t-1}$ measurable random vector $\theta_{t-1}(s_t) \in \mathbb{R}^n$ (Girsanov kernel, see [5]) has the following representation

$$\theta_{t-1}(s_t) = \Delta_0(s_t)\psi_t + \Delta_1(s_t)y_{t-1} + \dots + \Delta_p(s_t)y_{t-p}, \quad t = 1, \dots, T, \tag{2.4}$$

where $\Delta_0(s_t)$ is an $(n \times k)$ random coefficient matrix and $\Delta_i(s_t)$, $i = 1, \dots, p$ are $(n \times n)$ random coefficient matrices, which are measurable with respect to the σ–field $\mathcal{I}_{t-1}$. In order to change from the real probability measure $\mathbb{P}$ to some risk–neutral probability measure $\tilde{\mathbb{P}}$, for the random vectors $\theta_{t-1}(s_t)$, we define the following state price density process:

$$L_t \mid \mathcal{F}_0 := \prod_{m=1}^{t} \exp\bigg\{\theta_{m-1}^T(s_m)\Sigma^{-1}(s_m)\big(y_m - \Pi(s_m)\mathsf{Y}_{m-1}\big) - \frac{1}{2}\theta_{m-1}^T(s_m)\Sigma^{-1}(s_m)\theta_{m-1}(s_m)\bigg\}$$

for $t = 1, \dots, T$. Then it can be shown that $\{L_t\}_{t=1}^T$ is a martingale with respect to the filtration $\{\mathcal{H}_t\}_{t=1}^T$ and the real probability measure $\mathbb{P}$. So $\mathbb{E}[L_T|\mathcal{H}_0] = \mathbb{E}[L_1|\mathcal{H}_0] = 1$.

In order to formulate the following Theorem which is a trigger of option pricing with MS–VAR process and will be used in the rest of the paper, we define following matrices and vector:

$$\Psi(s) := \begin{bmatrix} I_n & 0 & \dots & 0 & 0 \\ -A_1(s_2) - \Delta_1(s_2) & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \\ 0 & 0 & \dots & -A_1(s_T) - \Delta_1(s_T) & I_n \end{bmatrix}$$

$\bar{\Sigma}(s) := \operatorname{diag}\{\Sigma(s_1), \dots, \Sigma(s_T)\}$, $\alpha(s) := (\alpha(s_1), \dots, \alpha(s_T)^T)^T$, and

$$\delta(s) := \begin{bmatrix} \big(A_0(s_1) + \Delta_0(s_1)\big)\psi_1 + \big(A_1(s_1) + \Delta_1(s_1)\big)y_0 + \dots + \big(A_p(s_1) + \Delta_p(s_1)\big)y_{1-p} \\ \big(A_0(s_2) + \Delta_0(s_2)\big)\psi_2 + \big(A_2(s_2) + \Delta_2(s_2)\big)y_0 + \dots + \big(A_p(s_2) + \Delta_p(s_2)\big)y_{2-p} \\ \vdots \\ \big(A_0(s_{T-1} + \Delta_0(s_{T-1})\big)\psi_{T-1} \\ \big(A_0(s_T) + \Delta_0(s_T)\big)\psi_T \end{bmatrix}.$$

Then, the following Theorem holds.

Theorem 1. *Let a MS–VAR process y_t is given by equation* (2.1) *or* (2.2)*, for $t = 1, \dots, T$, representation of a random vector $\theta_{t-1}(s_t)$ which is measurable with respect to σ–field $\mathcal{I}_{t-1}$ is given by equation* (2.4)*. We define the following new (risk–neutral) probability measure*

$$\tilde{\mathbb{P}}[A|\mathcal{F}_0] := \int_A L_T(\omega|\mathcal{F}_0)d\mathbb{P}[\omega|\mathcal{F}_0] \quad \text{for all } A \in \mathcal{H}_T.$$

Let

$$\delta(s) = \begin{bmatrix} \bar{\delta}_1(\bar{s}_t) \\ \bar{\delta}_2(\bar{s}_t^c) \end{bmatrix}, \quad \Psi(s) = \begin{bmatrix} \Psi_{11}(\bar{s}_t) & 0 \\ \Psi_{21}(\bar{s}_t^c) & \Psi_{22}(\bar{s}_t^c) \end{bmatrix} \quad \text{and} \quad \bar{\Sigma}(s) = \begin{bmatrix} \bar{\Sigma}_{11}(\bar{s}_t) & 0 \\ 0 & \bar{\Sigma}_{22}(\bar{s}_t^c) \end{bmatrix}$$

be partitions corresponding to random sub vectors $\bar{y}_t$ and $\bar{y}_t^c$ of a random vector $y = (y_1^T, \dots, y_T^T)^T$. Then the following probability laws hold:

$$y \mid \mathcal{H}_0 \sim \mathcal{N}\Big(\Psi(s)^{-1}\delta(s), \Psi(s)^{-1}\bar{\Sigma}(s)(\Psi(s)^{-1})^T\Big), \tag{2.5}$$

$$\bar{y}_t^c \mid \mathcal{H}_t \sim \mathcal{N}\Big(\Psi_{22}^{-1}(\bar{s}_t^c)\big(\bar{\delta}_2(\bar{s}_t^c) - \Psi_{21}(\bar{s}_t^c)\bar{y}_t\big), \Psi_{22}^{-1}(\bar{s}_t^c)\bar{\Sigma}_{22}(\bar{s}_t^c)(\Psi_{22}^{-1}(\bar{s}_t^c))^T\Big), \tag{2.6}$$

under the risk–neutral probability measure $\tilde{\mathbb{P}}$.

2.2. Log–normal System

Under MS–VAR framework, [4] introduced foreign–domestic market and obtained pricing formulas for frequently used options. Because the idea of domestic market can be used to domestic–foreign market, to simplify the calculation, here we will focus on a domestic market. We assume that financial variables, which are composed of a domestic log spot rate and domestic assets, and economic variables are together placed on MS–VAR process y_t. To extract the financial variables from the process y_t, we introduce the following vectors and matrices: $e_i := (0,\dots,0,1,0,\dots,0)^T \in \mathbb{R}^n$ is a unit vector, that is, its i-th component is 1 and others are zero, $M_1 := \big[I_{n_z} : 0_{n_z\times n_x}\big]$, and $M_2 := \big[0_{n_x\times n_z} : I_{n_x}\big]$.

Let r_t be a domestic spot interest rate. We define $\tilde{r}_t := \ln(1+r_t)$. Then $\tilde{r}_t$ represents total log return at time t and we will refer to it as log spot rate. Since the spot interest rate at time t is known at time $(t-1)$, we can assume that the log spot rate is placed on the 1st component of the process y_{t-1}. In this case, $\tilde{r}_t = e_1^T y_{t-1}$. Let $n_z \geq 1$ and $z_t := M_1 y_t$ be an $(n_z \times 1)$ vector at time t that include the domestic log spot rate. Since the first component of the process z_t corresponds to the domestic log spot rate, we assume that other components of the process z_t correspond to economic variables that affect the financial variables. So, the log spot rate is not constant and is explained by its own and other variables' lagged values in the VAR system y_t.

We further suppose that $\tilde{x}_t := \ln(x_t) = M_2 y_t$ is an $(n_x \times 1)$ log price process of the domestic assets, where x_t is an $(n_x \times 1)$ price process of the domestic assets. This means log prices of the domestic assets are placed on (n_z+1)–th to (n_z+n_x)–th components of the MS–VAR process y_t. As a result, the domestic market is given by the following system:

$$\begin{cases} z_t = \Pi_1(s_t)\mathsf{Y}_{t-1} + \zeta_t \\ \tilde{x}_t = \Pi_2(s_t)\mathsf{Y}_{t-1} + \eta_t \\ D_t = \exp\{-\tilde{r}_1 - \tilde{r}_2 - \dots - \tilde{r}_t\} = \frac{1}{\prod_{m=1}^t (1+r_m)} \\ \tilde{r}_t = e_1^T y_{t-1} \end{cases}, \quad t = 1,\dots,T, \tag{2.7}$$

where D_t is a domestic discount process, $\zeta_t := M_1\xi_t$ and $\eta_t := M_2\xi_t$ are residual processes of the processes z_t and $\tilde{x}_t$, respectively, $\Pi_1(s_t) := M_1\Pi(s_t)$ and $\Pi_2(s_t) := M_2\Pi(s_t)$ are random coefficient matrices. For the system, $D_t x_t$ represent a discounted price process of the domestic assets. If we define a random vector $\hat{\theta}_{2,t-1}(s_t) := M_2(y_{t-1} - \Pi(s_t)\mathsf{Y}_{t-1}) + i_{n_x} e_1^T y_{t-1}$, then it can be shown that

$$D_t x_t = \big(D_{t-1}x_{t-1}\big) \odot \exp\big(\eta_t - \hat{\theta}_{2,t-1}(s_t)\big), \tag{2.8}$$

where $\odot$ means the Hadamard product. The random vector $\hat{\theta}_{2,t-1}(s_t)$ which is measurable with respect to σ-field $\mathcal{I}_{t-1}$ can be represented by

$$\hat{\theta}_{2,t-1}(s_t) = \hat{\Delta}_0(s_t)\psi_t + \hat{\Delta}_1(s_t)y_{t-1} + \dots + \hat{\Delta}_p(s_t)y_{t-p}, \tag{2.9}$$

where $\hat{\Delta}_0(s_t) := -M_2A_0(s_t)$, $\hat{\Delta}_1(s_t) := M_2\big(I_n - A_1(s_t)\big) + i_{n_x}e_1^T$ and for $m = 2,\dots,T$, $\hat{\Delta}_m(s_t) := -M_2A_m(s_t)$. According to equation (2.8), as $D_{t-1}x_{t-1}$ is $\mathcal{H}_{t-1}$ measurable, in order to the discounted process $D_t x_t$ is a martingale with respect to the filtration $\mathcal{H}_t$ and some risk-neutral probability measure $\tilde{\mathbb{P}}$, we must require that

$$\tilde{\mathbb{E}}\big[\exp\big\{\eta_t - \hat{\theta}_{2,t-1}(s_t)\big\}\big|\mathcal{H}_{t-1}\big] = i_{n_x}, \tag{2.10}$$

where $\tilde{\mathbb{E}}$ denotes a expectation under the risk-neutral probability measure $\tilde{\mathbb{P}}$.

It should be noted that condition (2.10) corresponds only to the white noise process η_t. Thus, we need to impose a condition on the processes ζ_t under the risk–neutral probability measure. This condition is fulfilled by $\tilde{\mathbb{E}}[\exp\{\zeta_t\}|\mathcal{H}_{t-1}] = \hat{\theta}_{1,t}$ for $\mathcal{G}_{t-1}$ measurable any random variable $\hat{\theta}_{1,t}$. Because for any admissible choices of $\theta_{1,t}$, condition (2.10) holds, the market is incomplete. But prices of the options, which will be defined below are still consistent with the absence of arbitrage. For this reason, to price the options, we will use optimal Girsanov kernel process θ_t, which minimizes the variance of the state price density process and the relative entropy. According to [4], the optimal Girsanov kernel process is given by

$$\theta_t = \Theta_t(\hat{\theta}_{2,t} - \alpha_{2,t}), \quad t = 1, \dots, T,$$

where $\Theta_t := \big[(\Sigma_{12}(s_t)\Sigma_{22}^{-1}(s_t))^T : I_{n_x}\big]^T$ and $\alpha_{2,t} := \frac{1}{2}\mathcal{D}\big[\Sigma_{22}(s_t)\big]$. Here $\Sigma_{12}(s_t) := M_1\Sigma(s_t)M_2^T$ and $\Sigma_{22}(s_t) := M_2\Sigma(s_t)M_2^T$ and $\mathcal{D}[O]$ is a vector, whose elements consist of diagonal elements of a generic square matrix O.

We denote first column of a generic matrix O by $(O)_1$ and a matrix, which consists of other columns of the matrix O by $(O)_1^c$. Then, the representation of the Girsanov kernel process is

$$\theta_t = \Delta_{0,t}\psi_t + \Delta_{1,t}y_{t-1} + \dots + \Delta_{p,t}y_{t-p}, \quad t = 1, \dots, T,$$

where $(\Delta_{0,t})_1 = \Theta_t\big((\hat{\Delta}_{0,t})_1 - \alpha_{2,t}\big)$, $(\Delta_{0,t})_1^c = \Theta_t(\hat{\Delta}_{0,t})_1^c$, and for $m = 1, \dots, p$, $\Delta_{m,t} := \Theta_t\hat{\Delta}_{m,t}$. As a result, due to Theorem 1, conditional on $\mathcal{H}_t$, a distribution of the random vector $\bar{y}_t^c$ is given by

$$\bar{y}_t^c = (y_{t+1}^T, \dots, y_T^T)^T \mid \mathcal{H}_t \sim \mathcal{N}\big(\mu_{2.1}^\alpha(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}\big)$$

under a risk–neutral probability measure $\tilde{\mathbb{P}}$, where $\mu_{2.1}^\alpha(\bar{y}_t, \bar{s}_t^c) := \Psi_{22}^{-1}(\bar{s}_t^c)\big(\bar{\delta}_2(\bar{s}_t^c) - \Psi_{21}(\bar{s}_t^c)\bar{y}_t\big)$ and $\Sigma_{22.1}(\bar{s}_t^c) := \Psi_{22}^{-1}(\bar{s}_t^c)\bar{\Sigma}_{22}(\bar{s}_t^c)(\Psi_{22}^{-1}(\bar{s}_t^c))^T$ are mean vector and covariance matrix of the random vector $\bar{y}_t^c$ given $\mathcal{H}_t$, respectively.

Let $\tilde{x} := (\tilde{x}_1^T, \dots, \tilde{x}_T^T)^T$ be a log of a price vector $x := (x_1^T, \dots, x_T^T)$. Then in terms of y, the log of price process is represented by $\tilde{x} = (I_T \otimes M_2)y$. Now we introduce a vector that deals with the domestic risk-free spot interest rate: a vector $\gamma_{u,v}$ is defined by for $v > u$, $\gamma_{u,v}^T := \big[0_{1\times[(u-t)n]} : i_{v-u-1}^T \otimes e_1^T : 0_{1\times[(T-v+1)n]}\big]$ and for $v = u$, $\gamma_{u,v} := 0 \in \mathbb{R}^{[T-t]n}$. Then observe that for $t \leq u \leq v$,

$$\sum_{m=u+1}^{v} \tilde{r}_m = e_i^T y_u 1_{\{u<v\}} + \gamma_{u,v}^T \bar{y}_t^c.$$

According to [12], clever change of probability measure lead to significant reduction in computational burden of derivative pricing. Therefore, we will consider some probability measures, originated from the risk-neutral probability measure $\tilde{\mathbb{P}}$. In this and following sections, we will assume that $0 \leq t \leq u \leq T$. We define the following map defined on σ-field $\mathcal{H}_T$:

$$\tilde{\mathbb{P}}_{t,u}^i\big[A|\mathcal{H}_t\big] := \frac{1}{D_t x_{i,t}} \int_A D_u x_{i,u} d\tilde{\mathbb{P}}\big[\omega|\mathcal{H}_t\big], \quad \text{for all } A \in \mathcal{H}_T.$$

Because the discounted process $D_t x_t$ get positive values and for $0 \leq t \leq u \leq T$, $\tilde{\mathbb{E}}[D_u x_u|\mathcal{H}_t] = D_t x_t$ (as it is a martingale with respect to the filtration $\{\mathcal{H}_t\}_{t=1}^T$ and risk-neutral probability measure $\tilde{\mathbb{P}}$), the map become probability measure. If we define $\beta_{t,u}^i = (i_{u-t}^T, 0_{1\times(T-u)})^T \otimes e_{n_z+i}$, where $\otimes$ is the Kronecker product, then it can be shown that a conditional distribution of the random vector $\bar{y}_t^c$ is given by

$$\bar{y}_t^c = (y_{t+1}^T, \dots, y_T^T)^T \mid \mathcal{H}_t \sim \mathcal{N}\Big(\mu_{t,u}^i(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\Big),$$

under measure $\tilde{\mathbb{P}}_{t,u}^i$, where $\mu_{t,u}^i(\bar{y}_t, \bar{s}_t^c) := \mu_{2.1}^\alpha(\bar{y}_t, \bar{s}_t^c) + \Psi_{22}^{-1}\bar{\Sigma}_2(\bar{s}_t^c)\beta_{t,u}^i$ and $\Sigma_{22.1}(\bar{s}_t^c)$ are mean vector and covariance matrix of the random vector $\bar{y}_t^c$ given $\mathcal{H}_t$.

To price rainbow options and lookback options which will be appear the following sections we will use the following two Lemmas which are given in [4] and are direct extension of the results in [3].

Lemma 1. *For $t = 0, \dots, T-1$, the following relation holds*

$$f(\bar{s}_t^c|\mathcal{G}_t) := \tilde{\mathbb{P}}\big(\bar{s}_t^c = \bar{s}_t^c|\mathcal{G}_t\big) = \frac{f(\Pi, \Gamma|\bar{s}_t^c, \bar{s}_t, \mathcal{F}_0)\displaystyle\prod_{m=t+1}^{T} p_{s_{m-1}s_m}}{\displaystyle\sum_{\bar{s}_t^c} f(\Pi, \Gamma|\bar{s}_t^c, \bar{s}_t, \mathcal{F}_0)\prod_{m=t+1}^{T} p_{s_{m-1}s_m}}, \tag{2.11}$$

where $p_{s_0 s_1} := \mathbb{P}(s_1 = s_1|\mathsf{P}, \mathcal{F}_0)$ *and* $p_{s_{m-1}s_m} := \mathbb{P}(s_m = s_m|\mathsf{P}, s_{m-1} = s_{m-1}, \mathcal{F}_0)$ *for* $m = t+1, \dots, T$.

If we denote normal distribution function with mean μ and covariance matrix Ω at event A by $\mathcal{N}(A, \mu, \Omega)$, then it follows from Lemma 1 that for all $A \in \mathcal{H}_T$

$$\tilde{\mathbb{P}}_{t,u}^i[A|\mathcal{G}_t] = \sum_{\bar{s}_t^c} \mathcal{N}\big(A, \mu_{t,u}^i(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\big) f(\bar{s}_t^c|\mathcal{G}_t).$$

Let us denote conditional on a generic σ-field $\mathcal{F}$ joint density function of a generic random vector X by $\tilde{f}(X|\mathcal{F})$ under $\tilde{\mathbb{P}}$ and let $\mathcal{J}_t := \sigma(\bar{\Pi}_t) \vee \sigma(\bar{\Gamma}_t) \vee \sigma(\bar{s}_t) \vee \mathcal{F}_0$. Then the following Lemma is true.

Lemma 2. *Conditional on $\mathcal{F}_t$, joint density of $\bar{\Pi}_t$, $\bar{\Gamma}_t$, P, and $\bar{s}_t$ is given by*

$$\tilde{f}\big(\Pi, \Gamma, \mathsf{P}, \bar{s}_t|\mathcal{F}_t\big) = \frac{\tilde{f}(\bar{y}_t|\mathcal{J}_t)\Big(\displaystyle\sum_{\bar{s}_t^c} f(\Pi, \Gamma|\bar{s}_t, \bar{s}_t^c, \mathcal{F}_0)\prod_{m=t+1}^{T} p_{s_{m-1}s_m}\Big)\prod_{m=1}^{t} p_{s_{m-1}s_m} f(\mathsf{P}|\mathcal{F}_0)}{\displaystyle\sum_{\bar{s}_T}\int_{\Pi,\Gamma,\mathsf{P}} \tilde{f}(\bar{y}_t|\mathcal{J}_t) f(\Pi, \Gamma|\bar{s}_T, \mathcal{F}_0)\prod_{m=1}^{T} p_{s_{m-1}s_m} f(\mathsf{P}|\mathcal{F}_0) d\Pi d\Gamma d\mathsf{P}}, \tag{2.12}$$

for $t = 1, \dots, T$ with convention $p_{s_1} = p_{s_0 s_1}$, where for $t = 1, \dots, T$,

$$\tilde{f}(\bar{y}_t|\mathcal{J}_t) = c\exp\Big\{-\frac{1}{2}\big(\bar{y}_t - \mu_1^*(\bar{s}_t)\big)^T(\Sigma_{11}^*)^{-1}(\bar{s}_t)\big(\bar{y}_t - \mu_1^*(\bar{s}_t)\big)\Big\}$$

with $c := \frac{1}{(2\pi)^{nt/2}\prod_{m=1}^t |\Sigma_m(s_m)|^{1/2}}$, $\mu_1^*(\bar{s}_t) := \Psi_{11}^{-1}(\bar{s}_t)\bar{\delta}_1(\bar{s}_t)$ *is a mean vector, and* $\Sigma_{11}^*(\bar{s}_t) := \Psi_{11}^{-1}(\bar{s}_t)\bar{\Sigma}_{11}(\bar{s}_t)(\Psi_{11}^{-1}(\bar{s}_t))^T$ *is a covariance matrix.*

Now we present a Lemma, which is used to calculate expectation of a random variable $D_v/D_u 1_A$ with respect to a generic probability measures.

Lemma 3. *Let $\bar{y}_t^c \mid \mathcal{H}_t \sim \mathcal{N}\big(\mu^G(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\big)$ under a generic probability measure $\tilde{\mathbb{P}}^G$. Then, for $A \in \mathcal{H}_T$ and $t \vee u \leq v$, it holds*

$$\tilde{\mathbb{E}}^G\left[\frac{D_v}{D_u}1_A\middle|\mathcal{H}_t\right] = \frac{D_{t\vee u}}{D_u}\exp\big\{\big[a^G\big]_{t\vee u}^v(\bar{y}_t, \bar{s}_t^c)\big\}\mathcal{N}\big(A, \big[\mu^G\big]_{t\vee u}^v(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\big)$$

where for $t \leq u \leq v$,

$$\big[a^G\big]_u^v(\bar{y}_t, \bar{s}_t^c) = -e_1^T y_t 1_{\{u=t, v>u\}} - \gamma_{u,v}^T\mu^G(\bar{y}_t, \bar{s}_t^c) + \frac{1}{2}\gamma_{u,v}^T\Sigma_{22.1}(\bar{s}_t^c)\gamma_{u,v}$$

and $\big[\mu^G\big]_u^v(\bar{y}_t, \bar{s}_t^c) = \mu^G(\bar{y}_t, \bar{s}_t^c) - \Sigma_{22.1}(\bar{s}_t^c)\bar{R}_t^{cT}j_{u,v}$.

3. Rainbow Options

Rainbow options are usually calls or puts on the maximum or minimum of underlying assets. A number of assets is called a number of colors of a rainbow and each asset is referred to as a color of the rainbow. [26] introduced rainbow options with two assets. Its extension is given by [19] for rainbow options with more than two assets using multidimensional normal cumulative distribution functions. In this section, we will present pricing formulas of call and put options and lookback options on maximum and minimum of several asset prices which are without default risk. Here we impose weights on all underlying assets at all time period. Therefore, the options depart from existing rainbow and lookback options. To price the rainbow options and lookback options, we reconsider domestic market, which is given by equation (2.7). We define maximum and minimum of prices of the domestic assets:

$$\overline{M}_t := \max_{1\leq u\leq t}\{M_u\} \quad \text{and} \quad \overline{m}_t := \min_{1\leq u\leq t}\{m_u\}$$

for $t = 1, \dots, T$, where

$$M_u := \max_{1\leq i\leq n_x}\{w_{i,u}x_{i,u}\} \quad \text{and} \quad m_u := \min_{1\leq i\leq n_x}\{w_{i,u}x_{i,u}\} \tag{3.1}$$

with $w_{i,u}$ is weight at time u of i-th asset. One of choices of the weight vector correspond to reciprocal of the assets at time 0. In this case, $w_{i,t}x_{i,t} = x_{i,t}/x_{i,0}$ represents total return at time t of i-th domestic asset. To price the rainbow options and lookback options, it will be sufficient to consider the following call option on maximum

$$C_{t,w}^{\overline{M}_T}(K) := \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(\overline{M}_T - K\big)^+\Big|\mathcal{I}_t\Big],$$

where T is a time of the option expiration and K is a strike price of the option. Let us denote a discounted contingent claim of the option by $\overline{H}_T^1$, that is,

$$\overline{H}_T^1 := D_T\big(\overline{M}_T - K\big)^+.$$

To simplify notations, we define the following random variables: $Z_{i,u} := w_{i,u}x_{i,u}$ is a price at time u of $w_{i,u}$ unit of i-th asset. Then, for all $i = 1, \dots, n_x$ and $u = 1, \dots, T$, event $\{\overline{M}_T = Z_{i,u}\} \cap \{\overline{M}_T \geq K\}$ (which means $Z_{i,u}$ is maximum and the option on maximum expires in the money) holds if and only if event $A_{i,u} \cap B_{i,u}$ holds, where $B_{i,u} := \big\{Z_{i,u} \geq K\big\}$ and $A_{i,u} := A_{i,u,1} \cap A_{i,u,2}$ with

$$A_{i,u,1} := \Big\{Z_{i,u} \geq Z_{1,1}, \dots, Z_{i,u} \geq Z_{n_x,1}, \dots, Z_{i,u} \geq Z_{1,t}, \dots, Z_{i,u} \geq Z_{n_x,t}\Big\}$$

and

$$A_{i,u,2} := \Big\{Z_{i,u} \geq Z_{1,t+1}, \dots, Z_{i,u} \geq Z_{n_x,t+1}, \dots, Z_{i,u} \geq Z_{1,T}, \dots, Z_{i,u} \geq Z_{n_x,T}\Big\}.$$

It is clear that the discounted contingent claim of the call option on maximum can be represented by

$$\overline{H}_T^1 = \sum_{i=1}^{n_x}\sum_{u=1}^{T} D_T\big(Z_{i,u} - K\big)1_{E_{i,u}}. \tag{3.2}$$

where $E_{i,u} := A_{i,u} \cap B_{i,u}$. Since for $1 \leq u \leq t$, random variables $Z_{i,u}$ are known at time t, the sets $A_{i,u,1}$ and $B_{i,u}$ must be represented by $A_{i,u,1} = B_{i,u} = \{\emptyset, \Omega\}$. Therefore, it allows us to deduce that

$$E_{i,u} = A_{i,u} \cap B_{i,u} \begin{cases} \in \big\{\emptyset, A_{i,u,2}\big\}, & \text{if} \quad 1 \leq u \leq t \\ = A_{i,u,2} \cap \big\{Z_{i,u} \geq \gamma\big\}, & \text{if} \quad t < u \leq T, \end{cases} \tag{3.3}$$

where $\gamma := \max\{\overline{M}_t, K\}$. Because for $1 \leq u \leq t$, $Z_{i,u}$ is known at time t and for $i = 1, \dots, n_x$, $\mu_{2.1}^\alpha(\bar{y}_t, \bar{s}_t^c) = \mu_{t,t}^i(\bar{y}_t, \bar{s}_t^c)$, due to Lemma 3, one obtain that conditional on $\mathcal{H}_t$ price at time t of the option on maximum is given by

$$\begin{aligned} C_{t,w}^{\overline{M}_T}(\mathcal{H}_t, K) = & \sum_{i=1}^{n_x}\sum_{u=1}^{T} w_{i,u}x_{i,u\wedge t}\exp\big\{[a_{t,u\vee t}^i]_{u\vee t}^T(\bar{y}_t, \bar{s}_t^c)\big\} \\ & \times \mathcal{N}\big(E_{i,u}, [\mu_{t,u\vee t}^i]_{u\vee t}^T(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\big) \\ & - K\sum_{i=1}^{n_x}\sum_{u=1}^{T}\exp\big\{[a_{2.1}^\alpha]_t^T(\bar{y}_t, \bar{s}_t^c)\big\}\mathcal{N}\big(E_{i,u}, [\mu_{2.1}^\alpha]_t^T(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\big), \end{aligned} \tag{3.4}$$

where for any real numbers $a, b \in \mathbb{R}$, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. In terms of the random log price vector $\bar{\tilde{x}}_t^c$ the set $A_{i,u,2}$ is expressed by

$$A_{i,u,2} = \big\{\tilde{L}_{i,u}\bar{\tilde{x}}_t^c \leq \tilde{b}_{i,u}\big\} \tag{3.5}$$

for $1 \leq u \leq t$ and $i = 1, \dots, n_x$, where $\tilde{b}_{i,u} := \big(\ln(Z_{i,u}/w_{1,t+1}), \dots, \ln(Z_{i,u}/w_{n_x,T})\big)^T$ and $\tilde{L}_{i,u} := I_{[T-t]n_x}$. Now we consider second line of equation (3.3). To represent the set $A_{i,u,2} \cap \big\{Z_{i,u} \geq \gamma\big\}$ in terms of the log price vector $\bar{\tilde{x}}_t^c$, we define the following matrix and vector:

$$L_{i,u} := \begin{bmatrix} I_{[u-t-1]n_x+i-1} & -i_{[u-t-1]n_x+i-1} & 0 \\ 0 & -i_{[T-u+1]n_x-i} & I_{[T-u+1]n_x-i} \\ 0 & -1 & 0 \end{bmatrix},$$

and

$$b_{i,u}^\gamma := \left(\ln\left(\frac{w_{i,u}}{w_{1,t+1}}\right), \dots, \ln\left(\frac{w_{i,u}}{w_{i-1,u}}\right), \ln\left(\frac{w_{i,u}}{w_{i+1,u}}\right), \dots, \ln\left(\frac{w_{i,u}}{w_{n_x,T}}\right), \ln\left(\frac{w_{i,u}}{\gamma}\right)\right)^T.$$

For the matrix $L_{i,u}$, its last row corresponds to the event $\big\{Z_{i,u} \geq \gamma\big\}$ and other rows correspond to the event $A_{i,u,2}$. In this case, we can deduce that

$$A_{i,u,2} \cap \big\{Z_{i,u} \geq \gamma\big\} = \big\{L_{i,u}\bar{\tilde{x}}_t^c \leq b_{i,u}^\gamma\big\} \tag{3.6}$$

for $u < t \leq T$ and $i = 1, \dots, n_x$. Let us introduce a simple Lemma, which will be used to price the call option on maximum.

LEMMA 4. *Let* $\bar{y}_t^c \mid \mathcal{H}_t \sim \mathcal{N}\big(\mu^G(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\big)$ *under a generic probability measure* $\tilde{\mathbb{P}}^G$. *Then for all* $\mathsf{A} \in \mathbb{R}^{k\times[(T-t)n_x]}$ *matrices, it holds*

$$\mathsf{A}\bar{\bar{x}}_t^c \mid \mathcal{H}_t \sim \mathcal{N}\Big(\mu^G(\bar{y}_t, \bar{s}_t^c, \mathsf{A}), \Sigma_{22.1}(\bar{s}_t^c, \mathsf{A})\Big)$$

under the generic probability measure $\tilde{\mathbb{P}}^G$, *where* $\mu^G(\bar{y}_t, \bar{s}_t^c, \mathsf{A}) := \mathsf{A}(I_{T-t}\otimes M_2)\mu^G(\bar{y}_t, \bar{s}_t^c)$ *and* $\Sigma_{22.1}(\bar{s}_t^c, \mathsf{A}) := \mathsf{A}(I_{T-t}\otimes M_2)\Sigma_{22.1}(\bar{s}_t^c)(I_{T-t}\otimes M_2^T)\mathsf{A}^T$.

Due to equations (3.3), (3.5) and (3.6), we have

$$\tilde{\mathbb{P}}^G\big[E_{i,u}\big|\mathcal{H}_t\big] = \begin{cases}\tilde{\mathbb{P}}^G\big[\tilde{L}_{i,u}\bar{\bar{x}}_t^c \leq \tilde{b}_{i,u}\big|\mathcal{H}_t\big]1_{A_{i,u,1}\cap B_{i,u}}, & \text{if} \quad 1\leq u\leq t,\\ \tilde{\mathbb{P}}^G\big[L_{i,u}\bar{\bar{x}}_t^c \leq b_{i,u}^\gamma\big|\mathcal{H}_t\big], & \text{if} \quad t<u\leq T\end{cases}$$

under a generic probability measure $\tilde{\mathbb{P}}^G$. We assume that weighted price at time u_* of i_*-th asset is maximum value in the history of the weighted prices of all assets up to and including time t, that is, $\overline{M}_t = Z_{i_*,u_*}$. Let us denote a normal distribution function with mean μ and covariance matrix Σ at point x by $\mathcal{N}(x,\mu,\Sigma)$. Then, according to equation (3.4) and Lemma 4, we can obtain that for given information $\mathcal{G}_t$, price at time t of the call option on maximum is given by

$$\begin{aligned}\hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, \gamma\star) := \sum_{\bar{s}_t^c}\Bigg[&\sum_{i=1}^{n_x}\sum_{u=t+1}^{T} w_{i,u}x_{i,t}\exp\big\{[a_{t,u}^i]_u^T(\bar{y}_t, \bar{s}_t^c)\big\}\\ &\times\mathcal{N}\big(b_{i,u}^{\gamma\star}, [\mu_{t,u}^i]_u^T(\bar{y}_t, \bar{s}_t^c, L_{i,u}^\star), \Sigma_{22.1}(\bar{s}_t^c, L_{i,u}^\star)\big)\\ &-K\sum_{i=1}^{n_x}\sum_{u=t+1}^{T}\exp\big\{[a_{2.1}^\alpha]_t^T(\bar{y}_t, \bar{s}_t^c)\big\}\\ &\times\mathcal{N}\big(b_{i,u}^{\gamma\star}, [\mu_{2.1}^\alpha]_t^T(\bar{y}_t, \bar{s}_t^c, L_{i,u}^\star), \Sigma_{22.1}(\bar{s}_t^c, L_{i,u}^\star)\big)\Bigg]f(\bar{s}_t^c|\mathcal{G}_t) + W_{i_*,u_*}.\end{aligned}\tag{3.7}$$

where $L_{i,u}^\star = L_{i,u}$, $b_{i,u}^{\gamma\star} = b_{i,u}^\gamma$ and

$$\begin{aligned}W_{i_*,u_*} := 1_{B_{i_*,u_*}}\sum_{\bar{s}_t^c}\Bigg[&\Big(w_{i_*,u_*}x_{i_*,u_*} - K\Big)\exp\big\{[a_{2.1}^\alpha]_t^T(\bar{y}_t, \bar{s}_t^c)\big\}\\ &\times\mathcal{N}\big(\tilde{b}_{i_*,u_*}^\star, [\mu_{2.1}^\alpha]_t^T(\bar{y}_t, \bar{s}_t^c, \tilde{L}_{i_*,u_*}^\star), \Sigma_{22.1}(\bar{s}_t^c, \tilde{L}_{i_*,u_*}^\star)\big)\Bigg]f(\bar{s}_t^c|\mathcal{G}_t)\end{aligned}$$

with $B_{i_*,u_*} = \{Z_{i_*,u_*}\geq K\}$, $\tilde{L}_{i_*,u_*}^\star = \tilde{L}_{i_*,u_*}$, and $\tilde{b}_{i_*,u_*}^\star = \tilde{b}_{i_*,u_*}$. We refer to the term W_{i_*,u_*} as tail term of the call option on maximum. Therefore, due to Lemmas 1 and 2, and the tower property of conditional expectation, price at time t of the call option on maximum with maturity T and strike price K is obtained by

$$\begin{aligned}C_{t,w}^{\overline{M}_T}(K) &= \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(\overline{M}_T - K\big)^+\Big|\mathcal{F}_t\Big] = \tilde{\mathbb{E}}\big[\hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, \gamma)\big|\mathcal{F}_t\big]\\ &= \sum_{\bar{s}_t}\int_{\Pi,\Gamma,\mathsf{P}} C_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, \gamma)\tilde{f}(\Pi, \Gamma, \mathsf{P}, \bar{s}_t|\mathcal{F}_t)d\Pi d\Gamma d\mathsf{P}\end{aligned}$$

Because in similar manner we can price other options using Lemmas 1 and 2, it is sufficient to price the options for the information $\mathcal{G}_t$. Now we list some option pricing formulas given $\mathcal{G}_t$, which are originated from above formula (3.7) corresponding to the call option on maximum of the domestic asset prices.

1. Let weighted price at time u_* of i_*-th asset be a maximum value in the history of the weighted prices of all assets up and including to time t. Then, conditional on information $\mathcal{G}_t$ price at time t of the call option on maximum with strike price K and expiration time T is given by
$$C_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K) := \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(\overline{M}_T - K\big)^+ \ \Big|\ \mathcal{G}_t\Big] = \hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, \gamma),$$
where input parameters of equation (3.7) are $B_{i_*,u_*} = \{Z_{i_*,u_*} \geq K\}$, $\tilde{L}_{i_*,u_*}^\star = \tilde{L}_{i_*,u_*}$, $\tilde{b}_{i_*,u_*}^\star = \tilde{b}_{i_*,u_*}$, $L_{i,u}^\star = L_{i,u}$ and $b_{i,u}^{\gamma\star} = b_{i,u}^\gamma$ with $\gamma = \overline{M}_t \vee K$.
2. Let weighted price at time u_* of i_*-th asset be a maximum value in the history of the weighted prices of all assets up to and including time t. Then, conditional on information $\mathcal{G}_t$ price at time t of a put option on maximum with strike price K and expiration time T is given by
$$\begin{aligned} P_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K) &:= \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(K - \overline{M}_T\big)^+ \ \Big|\ \mathcal{G}_t\Big] \\ &= \begin{cases} \hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, K) - \hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, \overline{M}_t) + W_{i_*,u_*} & \text{if} \quad \overline{M}_t \leq K, \\ 0 & \text{if} \quad \overline{M}_t > K, \end{cases} \end{aligned}$$
where input parameters of equation (3.7) are $B_{i_*,u_*} = \{Z_{i_*,u_*} \leq K\}$, $\tilde{L}_{i_*,u_*}^\star = \tilde{L}_{i_*,u_*}$, $\tilde{b}_{i_*,u_*}^\star = \tilde{b}_{i_*,u_*}$, $L_{i,u}^\star = L_{i,u}$, $b_{i,u}^{K\star} = b_{i,u}^K$ and $b_{i,u}^{\overline{M}_t\star} = b_{i,u}^{\overline{M}_t}$.
3. Let weighted price at time u_* of i_*-th asset is minimum value in the history of the weighted prices of all assets up to and including time t. Then, conditional on information $\mathcal{G}_t$ price at time t of a call option on minimum with strike price K and expiration time T is given by
$$\begin{aligned} C_{t,w}^{\overline{m}_T}(\mathcal{G}_t, K) &:= \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(\overline{m}_T - K\big)^+ \ \Big|\ \mathcal{G}_t\Big] \\ &= \begin{cases} \hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, \overline{m}_t) - \hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, K) + W_{i_*,u_*} & \text{if} \quad \overline{m}_t \geq K, \\ 0 & \text{if} \quad \overline{m}_t < K, \end{cases} \end{aligned}$$
where input parameters of equation (3.7) are $B_{i_*,u_*} = \{Z_{i_*,u_*} \geq K\}$, $\tilde{L}_{i_*,u_*}^\star = -\tilde{L}_{i_*,u_*}$, $\tilde{b}_{i_*,u_*}^\star = -\tilde{b}_{i_*,u_*}$, $L_{i,u}^\star = -L_{i,u}$, $b_{i,u}^{K\star} = -b_{i,u}^K$ and $b_{i,u}^{\overline{m}_t\star} = -b_{i,u}^{\overline{m}_t}$.
4. Let weighted price at time u_* of i_*-th asset is minimum value in the history of the weighted prices of all assets up to and including time t. Then, conditional on information $\mathcal{G}_t$ price at time t of a put option on minimum with strike price K and expiration time T is given by
$$P_{t,w}^{\overline{m}_T}(\mathcal{G}_t, K) := \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(K - \overline{m}_T\big)^+ \ \Big|\ \mathcal{G}_t\Big] = \hat{C}_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K, \gamma),$$
where input parameters of equation (3.7) are $B_{i_*,u_*} = \{Z_{i_*,u_*} \leq K\}$, $\tilde{L}_{i_*,u_*}^\star = -\tilde{L}_{i_*,u_*}$, $\tilde{b}_{i_*,u_*}^\star = -\tilde{b}_{i_*,u_*}$, $L_{i,u}^\star = -L_{i,u}$, and $b_{i,u}^{\gamma\star} = -b_{i,u}^\gamma$ with $\gamma = \overline{m}_t \wedge K$.
5. According to above formula for the call option on maximum, conditional on information $\mathcal{G}_t$ price at time t of a lookback call option with expiration time T is given by
$$L_{t,w}^C(\mathcal{G}_t) := \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(\overline{M}_T - M_T\big) \ \Big|\ \mathcal{G}_t\Big] = C_{t,w}^{\overline{M}_T}(\mathcal{G}_t, 0) - C_{t,\overline{w}}^{\overline{M}_T}(\mathcal{G}_t, 0)$$

where for $i = 1, \dots, n_x$, $\bar{w}_{i,T} := w_{i,T}$ and rest of the components of a vector $\bar{w}$ are zero.

6. According to above formula for the call option on minimum, conditional on information $\mathcal{G}_t$ price at time t of a lookback put option with expiration time T is given by

$$L_{t,w}^P(\mathcal{G}_t) := \frac{1}{D_t}\tilde{\mathbb{E}}\Big[D_T\big(m_T - \overline{m}_T\big) \ \Big| \ \mathcal{G}_t\Big] = C_{t,\bar{w}}^{\overline{m}_T}(\mathcal{G}_t, 0) - C_{t,w}^{\overline{m}_T}(\mathcal{G}_t, 0),$$

where $i = 1, \dots, n_x$, $\bar{w}_{i,T} := w_{i,T}$ and rest of the components of a vector $\bar{w}$ are zero.

It should be noted that if we know distribution of a random vector $\text{vec}(\Pi, \Gamma, \mathsf{P}, \bar{s}_t)$ conditional on $\mathcal{F}_t$, then one can price options by Monte–Carlo simulation methods. Let us illustrate an option pricing method using Monte–Carlo methods for the call option on maximum. To price the option by Monte–Carlo methods, first, we generate a sufficiently large number of random realizations $V_{t*} := (\Pi_*, \Sigma_*, \mathsf{P}_*, \bar{s}_{t*})$ from $f(\Pi, \Sigma, \mathsf{P}, \bar{s}_t|\mathcal{F}_t)$. Then we substitute them into the price formula of call option on maximum, $C_{t,w}^{\overline{M}_T}(\mathcal{G}_t, K)$ obtain a large number of $C_{t,w}^{\overline{M}_T}(V_{t*})$s. Finally, we average $C_{t,w}^{\overline{M}_T}(V_{t*})$s. By the law of large numbers, the average converges to theoretical option price $C_{t,w}^{\overline{M}_T}(K)$. This simulation method is better than a simulation method which is based on realizations from $f(\bar{y}_t^c, \Pi, \Gamma, \mathsf{P}, \bar{s}_t|\mathcal{F}_t)$, because the former one has lower variance than the last one.

To make statistical inference about the parameter vector conditional on the information $\mathcal{F}_t$, one may use the Gibbs sampling method, which generates a dependent sequence of parameters. In the Bayesian statistics, the Gibbs sampling is often used when the joint distribution is not known explicitly or is difficult to sample from directly, but the conditional distribution of each variable is known and is easy to sample from. Very simple explanation of the Gibbs sampling can be found in [7], which is mainly focused on marginal distribution. Monte–Carlo methods using the Gibbs sampling of MS–VAR process are proposed by authors. In particular, Gibbs sampling method of MS–AR(p) process is provided by [1] and its multidimensional extension is given by [20].

Note that using the idea in [4] one can obtain similar pricing formulas that correspond to rainbow options and lookback options of foreign asset prices and foreign currencies.

4. Locally Risk-Minimizing Strategy

[11] introduced the concept of mean–self–financing and extended the concept of complete market into incomplete market. If a discounted cumulative cost process is a martingale, then a portfolio plan is called mean-self-financing. In discrete time case, [10] developed a locally risk-minimizing strategy and obtained a recurrence formula for optimal strategy. According to [24] (see also [9]), under a martingale probability measure the locally risk-minimizing strategy and remaining conditional risk-minimizing strategy are same. In this section, we will consider the locally risk-minimizing strategy for the call option on maximum. In an insurance industry, for continuous time unit–linked term life and pure endowment insurances with guarantee, locally risk-minimizing strategies are obtained by [22].

To simplify notations we define: for $t = 1, \dots, T$, $\overline{X}_t := (\overline{X}_{1,t}, \dots, \overline{X}_{n_x,t})^T$ is a discounted price vector at time t and $\Delta\overline{X}_t := \overline{X}_t - \overline{X}_{t-1}$ is a difference vector at time t of the price vectors, where $\overline{X}_{i,u} := D_u x_{i,u}$ is a discounted price at time u of i-th asset. Note that $\Delta\overline{X}_t$ is a martingale difference with respect to the filtration $\{\mathcal{H}_t\}_{t=0}^T$

and risk-neutral measure $\tilde{\mathbb{P}}$. Following the idea in [9] and [10], one can obtain that for the filtration $\{\mathcal{F}_t\}_{t=0}^T$ and a generic discounted contingent claim $\overline{H}_T$, under risk-neutral measure $\tilde{\mathbb{P}}$ locally risk-minimizing strategy (h^0, h) is given by the following equations:

$$h_{t+1} = \Omega_{t+1}^{-1}\Lambda_{t+1} \quad \text{and} \quad h_{t+1}^0 = V_{t+1} - h_{t+1}^T X_{t+1} \tag{4.1}$$

for $t = 0, \dots, T-1$, where, $\Omega_{t+1} := \mathbb{E}\big[\Delta\overline{X}_{t+1}\Delta\overline{X}_{t+1}^T\big|\mathcal{F}_t\big]$, $\Lambda_{t+1} := \widetilde{\text{Cov}}\big[\Delta\overline{X}_{t+1}, \overline{H}_T\big|\mathcal{F}_t\big]$ and $\overline{V}_{t+1} := \tilde{\mathbb{E}}[\overline{H}_T|\mathcal{F}_{t+1}]$ for a square integrable random variable $\overline{H}_T$. It should be noted that since all the options are originated from the call option on maximum of several asset prices, it will be sufficient to consider locally risk–minimizing strategies that correspond to the call option on maximum. Because the difference of discounted price process $\Delta\overline{X}_t$ is a martingale difference with respect to the risk-neutral probability measure $\tilde{\mathbb{P}}$ and filtration $\{\mathcal{H}_t\}_{t=0}^T$, it follows that

$$\Lambda_{t+1} = \tilde{\mathbb{E}}\big[\overline{H}_T\overline{X}_{t+1}\big|\mathcal{F}_t\big] - \overline{V}_t\overline{X}_t. \tag{4.2}$$

For product of discounted price at time u of i-th asset and discounted price at time s of j-th asset, it can be shown that for $i, j = 1, \dots, n_x$ and $t \leq u, v$,

$$\tilde{\mathbb{E}}\big[\overline{X}_{i,u}\overline{X}_{j,v}|\mathcal{H}_t\big] = \overline{X}_{i,t}\overline{X}_{j,t}\exp\big\{\beta_{t,u}^{iT}\bar{\Sigma}_2(\bar{s}_t^c)\beta_{t,v}^j\big\} = \overline{X}_{i,t}\overline{X}_{j,t}\exp\bigg\{\sum_{m=t+1}^{u\wedge v}\sigma_{ij,m}(s_m)\bigg\}, \tag{4.3}$$

where $\sigma_{ij,m}(s_m)$ is (i,j)-th element of the random matrix at regime s_m, $\Sigma_m(s_m)$. Therefore, as $\overline{X}_t$ is a martingale with respect to filtration $\{\mathcal{H}_t\}_{t=0}^T$ and risk-neutral measure $\tilde{\mathbb{P}}$, equation (4.3) allows us to conclude that for $i, j = 1, \dots, n_x$, (i,j)-th element of the random matrix Ω_{t+1} is given by

$$\omega_{ij,t+1} = \tilde{\mathbb{E}}\big[\Delta\overline{X}_{i,t+1}\Delta\overline{X}_{j,t+1}|\mathcal{I}_t\big] = \overline{X}_{i,t}\overline{X}_{j,t}\Bigg(\tilde{\mathbb{E}}\bigg[\sum_{s_{t+1}=1}^{N}\exp\big\{\sigma_{ij,t+1}(s_{t+1})\big\}p_{s_{t+1}}\bigg|\mathcal{I}_t\bigg] - 1\Bigg). \tag{4.4}$$

Due to equation (4.3), as $\overline{X}_{i,t}, \overline{X}_{j,t} > 0$, one can define the following new probability measure:

$$\tilde{\mathbb{P}}_{t,u,v}^{i,j}[A|\mathcal{H}_t] := \frac{\exp\big\{-\beta_{t,u}^{iT}\bar{\Sigma}_2(\bar{s}_t^c)\beta_{t,v}^j\big\}}{\overline{X}_{i,t}\overline{X}_{j,t}}\int_A \overline{X}_{i,u}\overline{X}_{j,v}\tilde{\mathbb{P}}[\omega|\mathcal{H}_t], \quad \text{for all } A \in \mathcal{H}_T.$$

It can be shown that conditional distribution of random vector $\bar{y}_t^c$ given $\mathcal{H}_t$ is given by

$$\bar{y}_t^c \mid \mathcal{H}_t \sim \mathcal{N}\Big(\mu_{t,u,v}^{i,j}(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\Big)$$

under probability measure $\tilde{\mathbb{P}}_{t,u,v}^{i,j}$, where $\mu_{t,u,v}^{i,j}(\bar{y}_t, \bar{s}_t^c) := \mu_{2.1}^{\alpha}(\bar{y}_t, \bar{s}_t^c) + \Psi_{22}^{-1}\bar{\Sigma}_2(\bar{s}_t^c)(\beta_{t,u}^i + \beta_{t,v}^j)$. In order to obtain locally risk-minimizing strategies that correspond to the call option on maximum, we need to calculate conditional expectations that have forms $\tilde{\mathbb{E}}[D_T\overline{X}_{j,v}1_A|\mathcal{H}_t]$, $\tilde{\mathbb{E}}[\overline{X}_{i,u}\overline{X}_{j,v}1_A|\mathcal{H}_t]$ and $\tilde{\mathbb{E}}[D_T/D_u\overline{X}_{i,u}\overline{X}_{j,v}1_A|\mathcal{H}_t]$ for a generic set $A \in \mathcal{H}_T$. It follows from the domestic and above probability measures and Lemma 3 that for $t \leq u, v$,

$$\tilde{\mathbb{E}}[D_u\overline{X}_{j,v}1_A|\mathcal{H}_t] = D_t\overline{X}_{j,t}\exp\big\{[a_{t,v}^j]_t^u(\bar{y}_t, \bar{s}_t^c)\big\}\mathcal{N}\big(A, [\mu_{t,v}^j]_t^u(\bar{y}_t, \bar{s}_t^c), \Sigma_{22.1}(\bar{s}_t^c)\big), \tag{4.5}$$

and

$$\tilde{\mathbb{E}}[D_T/D_u\overline{X}_{i,u}\overline{X}_{j,v}1_A|\mathcal{H}_t] = \overline{X}_{i,t}\overline{X}_{j,t}\exp\bigg\{\sum_{m=t+1}^{u\wedge v}\sigma_{ij,m}(s_m) + [a_{t,u,v}^{i,j}]_u^T(\bar{y}_t,\bar{s}_t^c)\bigg\} \times \mathcal{N}\big(A,[\mu_{t,u,v}^{i,j}]_u^T(\bar{y}_t,\bar{s}_t^c),\Sigma_{22.1}(\bar{s}_t^c)\big). \tag{4.6}$$

In terms of the discounted price process $\overline{X}_{i,t}$, the discounted contingent claim of the call option on maximum, $\overline{H}_T$, which is given by equation (3.2) can be represented by

$$\overline{H}_T^1 = \sum_{i=1}^{n_x}\sum_{u=1}^{t}D_T\big(w_{i,u}x_{i,u}-K\big)1_{E_{i,u}} + \sum_{i=1}^{n_x}\sum_{u=t+1}^{T}D_T/D_uw_{i,u}\overline{X}_{i,u}1_{E_{i,u}} - K\sum_{i=1}^{n_x}\sum_{u=t+1}^{T}D_T1_{E_{i,u}}.$$

To obtain Λ_{t+1} corresponding to the call option on maximum, we define $R_{j,t+1}(\mathcal{G}_t) := \tilde{\mathbb{E}}\big[\overline{H}_T\overline{X}_{j,t+1}|\mathcal{G}_t\big]$. Then, equations (4.5)-(4.6) and Lemma 1 allow us to conclude that the expectation is given by the following equations:

$$\begin{aligned}R_{j,t+1}(\mathcal{G}_t) = \sum_{\bar{s}_t^c}\Bigg[&\big(w_{i_*,u_*}x_{i_*,u_*}-K\big)D_t\overline{X}_{j,t}\exp\big\{[a_{t,t+1}^j]_t^T(\bar{y}_t,\bar{s}_t^c)\big\}\\ &\times\mathcal{N}\big(\tilde{b}_{i_*,u_*},[\mu_{t,t+1}^j]_t^T(\bar{y}_t,\bar{s}_t^c,\tilde{L}_{i_*,u_*}),\Sigma_{22.1}(\bar{s}_t^c,\tilde{L}_{i_*,u_*})\big)1_{B_{i_*,u_*}}\\ &+\sum_{i=1}^{n_x}\sum_{u=t+1}^{T}w_{i,u}\overline{X}_{i,t}\overline{X}_{j,t}\exp\big\{\sigma_{ij,t+1}(s_{t+1})+[a_{t,u,t+1}^{i,j}]_u^T(\bar{y}_t,\bar{s}_t^c)\big\}\\ &\times\mathcal{N}\big(b_{i,u}^\gamma,[\mu_{t,u,t+1}^{i,j}]_u^T(\bar{y}_t,\bar{s}_t^c,L_{i,u}),\Sigma_{22.1}(\bar{s}_t^c,L_{i,u})\big)\\ &-K\sum_{i=1}^{n_x}\sum_{u=t+1}^{T}D_t\overline{X}_{j,t}\exp\big\{[a_{t,t+1}^j]_t^T(\bar{y}_t,\bar{s}_t^c)\big\}\\ &\times\mathcal{N}\big(b_{i,u}^\gamma,[\mu_{t,t+1}^j]_t^T(\bar{y}_t,\bar{s}_t^c,L_{i,u}),\Sigma_{22.1}(\bar{s}_t^c,L_{i,u})\big)\Bigg]f(\bar{s}_t^c|\mathcal{G}_t).\end{aligned} \tag{4.7}$$

To simplify notations, let us introduce the following vector:

$$R_{t+1}(\mathcal{G}_t) := (R_{1,t+1}(\mathcal{G}_t),\dots,R_{n_x,t+1}(\mathcal{G}_t))^T.$$

Therefore, due to equations (4.2) and (4.7) one can obtain that for the call option on maximum, we have

$$\Lambda_{t+1} = \tilde{\mathbb{E}}\big[\overline{H}_T\overline{X}_{t+1}|\mathcal{F}_t\big] - \tilde{\mathbb{E}}\big[\overline{H}_T|\mathcal{F}_t\big]\overline{X}_t = \tilde{\mathbb{E}}\big[R_{t+1}(\mathcal{G}_t)|\mathcal{F}_t\big] - \overline{C}_{t,w}^{\overline{M}_T}(K)\overline{X}_t, \tag{4.8}$$

where $\overline{C}_{t,w}^{\overline{M}_T}(K) := D_tC_{t,w}^{\overline{M}_T}(K)$. As a result, if we substitute equations (4.4) and (4.8) into equation (4.1), we can obtain the locally risk–minimizing strategy for the call option on maximum of several asset prices.

5. Conclusion

Economic variables play important roles in any economic model, and sudden and dramatic changes exist in the financial market and economy. Therefore, in the paper, we introduced the MS–VAR process and obtained pricing and hedging formulas for the rainbow options and lookback options on maximum and minimum of several asset prices using the risk–neutral valuation method and locally risk–minimizing strategy.

It should be noted that the random MS–VAR process contains a simple VAR process, vector error correction model (VECM), BVAR, and MS–VAR process. To use our model, which is based on the MS–VAR process, as mentioned before one can use Monte–Carlo methods, see [20]. For the simple MS–VAR process, maximum likelihood methods are provided by [14–16] and [20] and for large BVAR process, we refer to [2]. To summarize, the main advantages of the paper are

– because we consider VAR process, the spot rate is not constant and is explained by its own and other variables' lagged values,
– it introduced economic variables, regime–switching, and heteroscedasticity to the options,
– it introduced the random MS–VAR process for valuation of the options, so the model will overcome over–parametrization,
– valuation and hedging of the options is not complicated,
– and the model contains simple VAR, VECM, BVAR, and MS–VAR processes.

References

1. Albert, James H., and Siddhartha Chib. 1993. "Bayes Inference via Gibbs Sampling of Autoregressive Time Series Subject to Markov Mean and Variance Shifts." Journal of Business & Economic Statistics 11(1): 1–15.
2. Bańbura, Marta., Giannone, Domenico., and Reichlin, Lucrezia. 2010. "Large Bayesian Vector Autoregressions." Journal of Applied Econometrics 25(1): 71–92.
3. Battulga, Gankhuu. 2022. "Stochastic DDM with regime–switching process." Numerical Algebra, Control and Optimization: 1–27
4. Battulga, Gankhuu. 2023. "Options Pricing under Bayesian MS–VAR Process." to be appear Numerical Algebra, Control and Optimization. Available at: `https://arxiv.org/abs/2109.05998`
5. Bjork, Tomas. 2009. "Arbitrage Theory in Continuous Time", 4th edition. Oxford University Press.
6. Black, Fischer., and Scholes, Myron. 1973. "The Pricing of Options and Corporate Liabilities." Journal of Political Economy 81(3): 637–654.
7. Casella, George., and George, Edward I. 1992. "Explaining the Gibbs sampler." The American Statistician 46(3): 167–174.
8. Doan, Thomas., and Litterman, Robert., and Sims, Christopher. "Forecasting and conditional projection using realistic prior distributions." Econometric reviews 3(1): 1–100.
9. Föllmer, Hans., and Schied, Alexander. 2004. "Stochastic Finance: An Introduction in Discrete Time", 2nd edition. Walter de Gruyter.
10. Föllmer, Hans., and Schweizer, Martin. 1989. "Hedging by sequential regression: An introduction to the mathematics of option trading." ASTIN Bulletin: The Journal of the IAA 18(2): 147–160.
11. Föllmer, Hans., and Sondermann, Dieter. 1986. "Hedging of non-redundant contingent claims." Contributions to mathematical economics: 206–223.
12. Geman, Helyette., and El Karoui, Nicole., and Rochet, Jean-Charles. 1995. "Changes of numeraire, changes of probability measure and option pricing." Journal of Applied Probability 32(2): 443–458.
13. Goldfeld, Stephen M., and Quandt, Richard E. 1973. "A Markov model for switching regressions." Journal of Econometrics 1(1): 3–15.
14. Hamilton, James D. 1989. "A new approach to the economic analysis of nonstationary time series and the business cycle." Econometrica: Journal of the Econometric Society: 357–384.
15. Hamilton, James D. 1990. "Analysis of time series subject to changes in regime." Journal of Econometrics 45(1–2): 39–70.
16. Hamilton, James D. 1994. "Time series econometrics." Princeton University Press.
17. Harrison, Micheal J., and Kreps, David M. 1979. "Martingales and arbitrage in multiperiod securities markets." Journal of Economic Theory 20(3): 381–408.
18. Harrison, Micheal J., and Pliska, Stanley R. 1981. "Martingales and stochastic integrals in the theory of continuous trading." Stochastic Processes and their Applications 11(3): 215–260.
19. Johnson, Herb. 1987. "Options on the Maximum or the Minimum of Several Assets." The Journal of Financial and Quantitative Analysis 22(3): 277–283.

20. Krolzig, Hans-Martin. 1997. ”Markov-switching vector autoregressions: Modelling, statistical inference, and application to business cycle analysis.” Springer Science & Business Media.
21. Merton, Robert C. 1973. ”Theory of Rational Option Pricing.” The Bell Journal of Economics and Management Science 4(1): 141-183.
22. Møller, Thomas. 1998. ”Risk-minimizing hedging strategies for unit-linked life insurance contracts.” ASTIN Bulletin: The Journal of the IAA 28(1): 17–47.
23. Quandt, Richard E. 1958. ”The estimation of the parameters of a linear regression system obeying two separate regimes.” Journal of the american statistical association 53(284): 873–880.
24. Schäl, Manfred. 1994. ”Mathematics of operations research.” On quadratic cost criteria for option hedging 19(1): 121–131.
25. Sims, Christopher A. 1980. ”Macroeconomics and Reality.” Econometrica 48(1): 1–48.
26. Stulz, RenéM. 1982. ”Options on the minimum or the maximum of two risky assets: analysis and applications.” Journal of Financial Economics 10(2): 161–185.
27. Tiao, George C., and Box, George EP. 1981. ”Modeling multiple time series with applications.” Journal of the American Statistical Association 76(376): 802–816.
28. Tong, Howell. 1983. ”Threshold models in non-linear time series analysis.” Springer Science & Business Media.
29. Zucchini, Walter., MacDonald, Iain L., and Langrock, Roland. 2016. ”Hidden Markov Models for Time Series: An Introduction Using R,” 2nd edition. CRC press.

Battulga Gankhuu
Applied Mathematics
National University of Mongolia
Ulaanbaatar, Mongolia.

battulga_gan@yahoo.com