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On some extremal problems in discrete geometry and applications of extremal graphs

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Abstract: Introducing the notions of distant pairs of vertices and big subtriangles of a polygon, and using extremal graph theory—specifically, Turan's graph—we establish upper bounds for:

- the sum of distances between all distant pairs of vertices in polygons with unit perimeter, and
- the sum of areas of all big subtriangles in convex polygons with unit area.

We also formulate a conjecture on the upper bound of the sum of the areas of all subtriangles in a convex polygon.

Key words: polygon, discrete geometry, extremal graph, Turan's graph, clique, partite graph

1. Introduction

The following problem was proposed by myself with a solution (see [1] page 46, page 112) to the 25th International Mathematical Olympiad:
IMO 1984, Problem 5: Let d be the sum of the lengths of all diagonals of a plane convex polygon with n vertices ($n > 3$) and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor - 2 \tag{1.1}$$

where [x] denotes the greatest integer not exceeding x.
Interestinly, this problem reappeared in 2007 as an open conjecture in [2], and was proved in [3]. Also in [4] proved that, the upper bound in (1) is true for any simple (sides of which not intersect except vertices) n-gons. (polygons with n vertices)

Repeating the above-mentioned proof (see.[1]) we will prove that the upper bound in (1) is true for any (not nessesery to be convex or simple) n-gons (in fact for any polygons in any metric space)
also, we will do a new approach to this problem using extremal graphs, namelly, the Turan's graphs. A pair of two vertices of a n-gon is called distant pair **if dissance** between them greater than $\frac{1}{3}$ of the perimeter of the n-gon, and we prove that, the maximum number of distant pairs is $[\frac{n}{2}][\frac{n+1}{2}]$ and the supremum of the sum of distances of *distant pairs* is the same as the supremum of the sum of distances between vertices of the polygon.

In the next part, we will consider instead of distants between vertices, areas of subtriangle (vertices of which are vertices of the polygon), and polygons will be only convex. A subtriangle is called *big-subtriangle* if its area is bigger than half of the area of the polygon.

We will prove the maximum number of the big subtriangles, is $\left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor$. Let $S^*(P)$ and $S(P)$ be sum of areas of all big-subtraingles and subtraingles respectively, of a convex n-gon P with unit area. Let M_n be the set of all convex n-gons with unit area Then we have

$$\sup_{P\in M_n} (S^*(P)) = \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor .$$

We have the following conjecture:
Conjecture 1

$$\sup_{P\in M_n} (S(P)) = \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor .$$

We proved this conjecture for $n \leq 6$.

2. Sum of Distances between Vertices of Polygons and Graph of Distant Pairs

In this part we will consider any n-gons with unit perimeter. Let $D(P)$ be the sum of distants between all vertices of the n-gon P Then, the inequality (1) formulates as follows.

$$\frac{n-1}{2} < D(P) < \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor \qquad (1')$$

Actually, the conjecture 2 in [2] was formulated that, the inequality (1') is true for convex n-gons with unit perimeter.

Theorem 1.1 For any n-gon with P unit perimeter next inequality is true:

$$D(P) < \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor \qquad (1'')$$

and

$$\sup (D(P)) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor$$

Proof (sf. [1], page 112) Let $A_1A_2\ldots A_n$ be a n-gon. $\sum_{i=1}^{n} |A_iA_{i+1}| = 1$ and indexes calculated mod(n).

Case Let $n = 2k$. Then $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] = k^2$. For all l and s with $2 \leq s \leq k-1$ we have by the triangle inequality rule (which is valid for any sequens of n-points $A_1, A_2, ..., A_n$in any metric space):

$$|A_lA_{l+s}| < |A_lA_{l+1}| + \cdots + |A_{l+s-1}A_{l+s}| \qquad (2.1)$$

and for all $m \leq n$

$$|A_mA_{m+k}| < \frac{p}{2} = \frac{1}{2}, \qquad (2.2)$$

where p is perimeter of P and $p = 1$. From (2) we have for each s with $2 \leq s \leq k-1$,

$$\sum_{l=1}^{2k} |A_lA_{l+s}| < \sum_{l=1}^{2K} (|A_lA_{l+1}| + \cdots + |A_{l+s-1}A_{l+s}|) = s\sum_{i=1}^{n} |A_iA_{i+1}| = s \cdot p = s \qquad (2.3)$$

Considering (2.2) and (2.3) we have

$$\sum_{s=2}^{k-1} \left(\sum_{l=1}^{2k} |A_lA_{l+s}| \right) + \sum_{m=1}^{K} |A_mA_{m+k}| \leq \sum_{s=2}^{k-1} sp + k \cdot \frac{p}{2} \qquad (2.4)$$

The left side of (2.4) is obviously the sum of diagonals, wich is $d = D(P) - p$ and the right side is $\frac{k^2-2}{2} \cdot p$. Therefore we have

$$D(P) - p < \frac{k^2-2}{2} p \quad \text{and so} \quad D(P) < \frac{1}{2} k^2 = \frac{1}{2} \left[\frac{n}{2}\right] \left[\frac{n+1}{2}\right]$$

Claim 2. $n = 2k - 1$. Then $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] = (k-1)k$. From (2.4) we wave

$$\sum_{s=2}^{K-1} \left(\sum_{l=1}^{2k} |A_l A_{l+s}| \right) < \sum_{s=2}^{k-1} sp \tag{2.5}$$

The last inequality (2.6) proves that $D(P) - d < \dfrac{k^2-k-2}{2} p$ and $B(P) < \dfrac{k^2-k}{2} = \frac{1}{2} \left[\frac{n}{2}\right] \left[\frac{n+1}{2}\right]$ and $(1'')$ has proved. In order to complete the proof we construct n-gon $A_1 A_2 \ldots A_n$ will unit perimeter as follows: $|A_1 A_n| = \left| A_{\left[\frac{n}{2}\right]} A_{\left[\frac{n}{2}\right]+1} \right| = \frac{1}{2} - \varepsilon$, where $|AB|$ is the distance between A and B, and

$$\sum_{i=1}^{\left[\frac{n}{2}\right]-1} |A_i A_{i+1}| = \sum_{j=\left[\frac{n}{2}\right]+1}^{n-1} |A_j A_{j+1}| = \varepsilon$$

We denote this n-gon with P_ε. Then $\lim_{\varepsilon \to 0} D(P_\varepsilon) = \frac{1}{2}\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ and the prove is completed.

Note. It is easy to see that the proof of the inequality ($1''$) valid for any n-gon in metric space thanks the triangle inequality rule. More over if the construction of the n gon P_ε is possible in this metric space (for example, it is possible in metric space $\mathbb{R}^k, k \geq 2$) the Theorem 1.1. is true in it.

Definition 1.2. a pair of vertices A and B of a polygon is called *distant pair* (abbr. *d.p.*) if

$$|AB| > \frac{p}{3}$$

where p is perimeter of the polygon.

Definition 1.3. Let $A_1 A_2 \ldots A_n$ be an n-gon. Then a graph $G(V, E)$ with vertices $V = \{A_1, A_2, \ldots, A_n\}$ and edges $E = \{A_i A_j \mid A_i, A_j$ - d.p.$\}$ is called the *graph of distant pairs* of the n-gon.

It is clear that G(V,E) does not contain any clique number 3.

Future, we need

Turan's Theorem [4]. Let n, r be integers and $r \geq 2$. amon all the graphs with n vertices and clique number less than r, the unique graph with the largest number of edges is the complete $r-1$ partite graph, where $r-1$ parts have sizes as nearly equal as possible.

For example, the largest number of edges is $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ for $r = 3$, and $\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right] + \left[\frac{n}{3}\right]\left[\frac{n+2}{3}\right] + + \left[\frac{n+1}{3}\right]\left\lfloor\frac{n+2}{3}\right\rfloor = \left[\frac{n^2}{3}\right]$ for $r = 4$.

Theorem 1.4. The maximum number of distant pairs of n-gon is

$$\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$$

Proof. As we noted the *graph of distant pairs* of n-gon does not contain clique number 3, therefore by Tutan's theorem the number of g.p. is less or equal with $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$. Other side, the n-gon P_ε which constructed during the proof of the Theorem 1.1. with enough small ε

has $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ distant pairs. The proof is completed.
Corollary 1.5. Let P_n be set of all n-gons with unit perimeter, $D^*(P)$ be sum of distances between all distant pairs of $P \in P_n$. Then

$$\sup_{P\in P_n} D(P) = \sup_{P\in P_n} D^*(P) = \frac{1}{2}\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$$

3. Sum of areas of subtriangles and Big-subtriangles of convex polygons and graph of good pairs

In this part we will consider only convex n-gons with unit area and we denote the set of all them by M_n. also, by $S(M)$ and $S^*(M)$ we denote the sum of areas of all subtriangles and big subtriangles of $M \in M_n$, respectively.
Proposition 2.1.

1. $\inf\limits_{M\in \mathrm{M}_n} S(M) = n-2$
2. $\sup\limits_{M\in \mathrm{M}_5} S(M) = 4$
3. $\sup\limits_{M\in \mathrm{M}_6} S(M) = 8$

We will omit the proof of this proposition.
Definition 2.2. A pair of vertices A and B of a polygon is called *good pair* (abbr. g.p.) if AB is a side of a *big-subtriangle* of the polygon.

Theorem 2.3. Let M be a convex n-gon ($n \geq 4$). Then among any four vertices of M at least two of them are not *good pair.*

The proof of this theorem will be given in part 4.

Definition 2.4. The *graph of good pairs* of a polygon M is defined as a graph $G(V,E)$, where the set of vertices V consists of vertices of M and $E = \{AB \mid A$ and B are *good pair in* $M\}$.
From the theorem 2.3 we have
Corallary 2.5 The *gaph of good pairs* of convex polygons does not contains clique number 4.
Theorem 2.6. Let G be a graph with n vertices ($n \geq 4$) , and does not contains clique number 4. Then the largest number of clique number 3 in the graph G is $[\frac{n}{3}][\frac{n+1}{3}][\frac{n+2}{3}]$. The unique graph with this number of clique number 3 is the complete 3 partite graph whose parts have sizes $[\frac{n}{3}]$, $[\frac{n+1}{3}]$ and $[\frac{n+2}{3}]$.
Proof. at first we will prove the next lemma:
Lemma 2.7. Let G be the graph in Theorem 2.6. Then G has a vertex with degree s such that

$$s \leq \left[\frac{n}{3}\right] + \left[\frac{n+1}{3}\right] \tag{3.1}$$

and consequently

$$\left[\frac{s}{2}\right]\left[\frac{s+1}{2}\right] \leq \left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right] \tag{3.2}$$

Proof of Lemma 2.7. Sum of degree of all vertices of graph $G(V,E)$ is $2|E|$. Therefore by the Turan's theorem

$$d(G) := \sum_{g\in G} deg(g) \leq 2\left[\frac{n^2}{3}\right] \tag{3.3}$$

If contrary, degree of each vertex equels or greater than $\left[\frac{n}{3}\right]+\left[\frac{n+1}{3}\right]+1$ then considering claims $n=3k+1$, $n=3k$, and $n=3k+1$ we can prove that $d(G)\geq n\left(\left[\frac{n}{3}\right]+\left[\frac{n+1}{3}\right]+1\right)>2\left[\frac{n^2}{3}\right]$, which is contradict with (3.3). The inequality (3.2) is a simple corollary of (3.1). Lemma has proved.

Now we will prove the Theorem 2.6 by induction with respect $n=|G|$.

For $n=4$ the Theorem is obvious.
Let the theorem 2.6. be true for n and $|G|=n+1$
By lemma 2.7. we can choose a vertex $A_1\in G$ with degree s, such that ((3.2) for $n+1$)

$$\left[\frac{s}{2}\right]\left[\frac{s+1}{2}\right]\leq\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right] \tag{3.4}$$

Let $A_2,\ldots,A_k$ be vertices of G not incident with A_1, and $B_1,\ldots,B_s$ be vertices of G incident with A_1. We consider two subgraphs of G :

$$G' := \{A_2,\ldots,A_k,B_1\ldots,B_s\} \text{ and}$$
$$G'' := \{B_1,\ldots,B_s\}.$$

Then $|G'|=n$ and, G' does not contains clique number 4 , G'' does not contains clique number 3 . Therefore, by induction G' has cliques number 3 not many as $\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$. Other side, cliques number 3 in which A_1 is a vertex are only thouse which have edge in G'', therefore number of them not greater than $\left[\frac{s}{2}\right]\left[\frac{s+1}{2}\right]$ by the Turan's theorem. Let $\Delta(G)$ be set of all cliques number 3 in the graph G. Then using inequality (3.4)

$$|\Delta(G)|\leq\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]+\left[\frac{s}{2}\right]\left[\frac{s+1}{2}\right]\leq$$
$$\leq\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]+\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]=\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]\left[\frac{n+3}{3}\right]$$

If $|\Delta(G)|=\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]\left[\frac{n+3}{3}\right]$ then as we seen

$$|\Delta(G')|=\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right] \text{ and } s=\left[\frac{n+1}{3}\right]+\left[\frac{n+2}{3}\right]. \text{ By induction}$$

G' is 3-partite with parts with sizes $\left[\frac{n}{3}\right]$, $\left[\frac{n+1}{3}\right]$ and $\left[\frac{n+2}{3}\right]$; G'' is 2- partite with parts with sizes $\left[\frac{n+1}{3}\right]$ and $\left[\frac{n+2}{3}\right]$. This says that A_1 belons in first parts of 3-partite graph G with $A_2,\ldots,A_k$ with size $\left[\frac{n}{3}\right]+1=\left[\frac{n+3}{3}\right]$, and the other parts coincide with the parts of G'', ie. have sizes $\left[\frac{n+1}{2}\right]$ and $\left[\frac{n+2}{3}\right]$. The proof of theorem 2.6 has completed.
Theorem 2.8. For any convex n-gon
1. Maximum number of *good pairs* is $\left[\frac{n^2}{3}\right]$
2. Maximum number of big *subtriangles*

$$\text{is } \left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$$

Proof By the Corollary 2.5. and theorem 2.6 it is enough to constract convex n-gon $A_1A_2....A_n$ with $\left[\frac{n^2}{3}\right]$ *good pairs* and with $\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$ *big-subtriangles.*
Let ABC be equilateral triangle with area $1+\sigma$. We take points A_n and A_1 in the side CA, A_i and A_{i+1} in the side AB, A_j and A_{j+1} in the side BC so that

$$|CA_n|=|A_1A|=|AA_i|=|A_{i+1}B|=|BA_j|=|A_{j+1},C|=\varepsilon;$$

where $i=\left[\frac{n}{3}\right], j=\left[\frac{n+1}{3}\right]+\left[\frac{n}{3}\right]$
As a result we have convex n-gon $A_1A_2\ldots A_iA_{i+1}\cdots A_jA_{j+1}\cdots A_n$, which we denote $N\varepsilon,\sigma$. When ε and σ are enough small $N_{\varepsilon\sigma}$, is the desired convex n-gon. The proof is finished.

Now, let M_n be set of all convex n-gons with unit area, and $S^*(M)$ be sum of areas of all *big-subtriangles* of the n-gon $M\in M_n$.

With the connection the conjecture 1 we will prove the next theorem:

Theorem 2.9. For any $n\geq 4$

$$\sup_{M\in M_n} S^*(M)=\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$$

Proof By the theorem 2.8 for any $M\in M_n$ $S^*(M)<\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$. On the other side, we have $\lim_{\varepsilon\to 0,\sigma\to 0} S^*(N_{\varepsilon\sigma})=\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$, where is $N_{\varepsilon,\sigma}$ - is the n-gon constructed in the proof of Theorem 2.8 and any time we can find ε and σ so that the aria of $N_{\varepsilon,\sigma}$ is unit. The proof has completed.
In relation to Theorem 2.9. believing that the analogy of the corollary 1.5 which is about the "distance", should be also true for "area", we have another formulation of our conjecture 1 as follows.

Conjecture $1'$

$$\sup_{M\in M_n} S(M)=\sup_{M\in M_n} S^*(M)$$

4. Proof of the Theorem 2.3.

In order to prove the Theorem 2.3, firstly we do some definitions and prove several lemmas.
D1. a subquadrilateral of a polygon is called *good subquadrilateral* (abbr: $g.s.q$) if each pair of its vertices is *good pair* (abbor:$g.p.$)
D.2 Let A and B be g.p. and a triangle ABN is bigsubtriangle, then N is called *good pair vertex* of g.p. of (A,B) (abb: $g.p.v.$ of (A,B))
D.3. Let R be a point, l be a line. Then the distans of R to the line l we denote as $d(l,R)$
D.4. Let ABCD be a subquadrilateral of a polygon. Then lines AB, BC, CD and DA divides the plane in 9 sections and the rest of vertices of the polygon locates in four of them. These sections called *next to the side-section*, for example *AB-side-section*.
D.5. For a subquadrilateral ABCD, if $d(DC,A)<d(\text{DC},B)$, then *BC-side-section* is called *wide section*, otherwise, *narrow section*.
D.6. Let K be $g.p.v.$ of (M,N). Then the line l with $K\in l$, parallel with MN is called the *level line* of K.

Lemma 3.1. If the triangles EAC and EBC of a pentagon ABCDE are *big subtriangles*, then the triangle ABD is not *big-subtriangles*.
Proof. Without losing the generality we can.Suppose that $d(AD,B)\leq d(AD,C)$. (see Fig. 1). Then, $S_{BPD}\leq S_{CDP}\leq S_{CDE}$ where S_{XYZ} is the area of the triangle XYZ. Also, $S_{ABP}\leq\max\{S_{ABC},S_{ABE}\}$ and $S_{ABD}=S_{BAP}+S_{BDP}$.

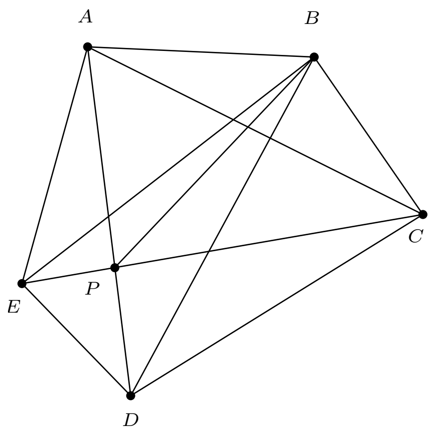

Figure 1: Fig:1

Claim 1. $S_{ABP} \leq S_{ABC}$. Then:

$$S_{ABD} \leq S_{ABC} + S_{CED} = S_{ABCDE} - S_{ACE} < \frac{1}{2} S_{ABCDE}.$$

Claim 2. $S_{ABP} \leq S_{ABE}$. Then:

$$S_{ABD} \leq S_{ABE} + S_{CED} = S_{ABCDE} - S_{BCE} < \frac{1}{2} S_{ABCDE}.$$

In both claims, it is proved that ABD is not large.

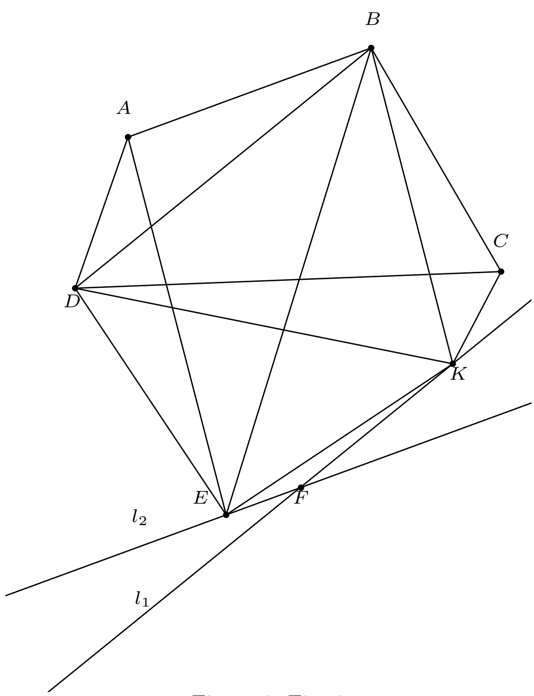

Figure 2: Fig. 2

Lemma 3.2. Let $A_1A_2A_3A_4$ be a $g.s.q$ of a polygon, and a $g.p.v.$ of the pair (A_1, A_3) exists in the side-section A_iA_{i+1}. Then in this section, $g.p.v.$ of the pair (A_{i+2}, A_{i+3}) does not exist, where the indices are calculated modulo 4.
Proof. Suppose, for the sake of contradiction, that the lemma is not true. Then we have a configuration as shown in Fig. 2, in which K is a g.p.v. of (D, B) in the DC-side-section, and E is a g.p.v. of (A, B), located in this section as well. Let l_1 and l_2 be the level lines of K and E, respectively, and let F be the intersection point of lines l_1 and l_2.

Claim 1. $d(DK, E) \geq d(DK, F)$.
If we change the vertex E by F , the area of the polygon decreases and F became a $g.p.v$ of (A,B). Now we change the vertex K by F, the area of the polygon decreases again and F is $g.p.v$ of (D,B), too. It is clear , that the $g.p.v$ of $g.p$(A,D),we note it by M, form pentagon $DABMF$ (in this sequence of vertices)in which triangles ABF and DBF are *big-subtriangles* (abbr *b.s.t.*) in reformed poligon with decreased area. Therefore by Lemma 3.1. the triangle DAM is not *b.s.t* in this polygon, therefore in the orginal polygon, which is contradict with the chois of the point M.

Claim.2. $d(DK, E) < d(DK, F)$. Then $d(DK, A) > d(DK, B)$ and therefore $S_{ADK} > S_{DKB}$ and this proves ADK is *b.s.t.* together with DBK consequently by the Lemma 3.1. thr triangle ABF is not *b.s.t.* , and this contradiction. The lemma has proved.

Lemma 3.3. $ABCD$ is $g.s.q.$ of a polygon. Then any $g.p.v.$ of diagonal pair (D, B) does not exists in *narrow section.*

Proof Let E be a $g.p.v$ of the diagonal pair (D, B) and located in contrary, in DA-*side -sector*, which is *narrow.* Then $d(AB, D) \leq d(AB, C)$, and therefore $d(EB, D) \leq d(EB, C)$, consequently $S_{EBD} \leq S_{EBC}$ and so, EBC is *b.s.t.* too. Now lemma 3.1 proves that the pair (D, C) could not to be *g. p.*, which is contradiction. The proof completed.

Lemma 3.4. If a $g.s.q.$ of a n-gon has more than two $g.p.v.$ in one section, then this $g.s.q.$ is also $g.s.q.$in a (n-1)-gon.

Proof. Let M, N and S are successive three $g.p.v.$and l_1, l_2 and l_3 are level lines of M, N and S respectively. (see. fig. 3)

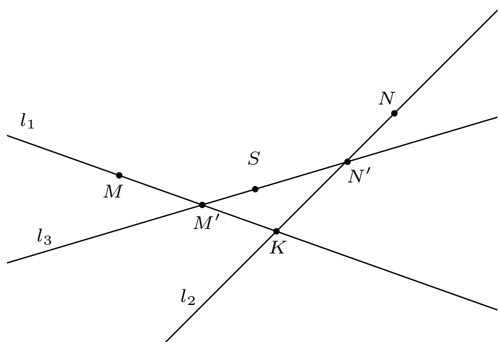

Figure 3: Fig. 3

Without losing the generality, we can suppose that $d(MN, M') \geq d(MN, N')$. We note that, if the n-gon has vertices between M and S or S and N. after elimination thiese vertices we will get $k - gon$ with $k < n$, in which the $g.s.q.$ is also $g.s .q.$ Now we change the vertex s with N' and get n-gon ... $MN'N$ ... instead of ... MSN ... and changing again the vertex N with N' we have $(n-1)$-gon $\ldots MN' \ldots$ in which the $g.s.q.$ is $g.s.q.$ too. The proof is completed.

Lemma 3.5. In any $g.\ s.\ q.$ two pairs with a common vertex could not have common $g.p.v.$

Proof: Let in contrary pairs (A, B) and (B, C) have common *g.p.v.* M. Then M does not locate in angle $\angle ABC$. Let N be *g.p.v.* of (A, C). Then N must be located as showing in fig. 4.

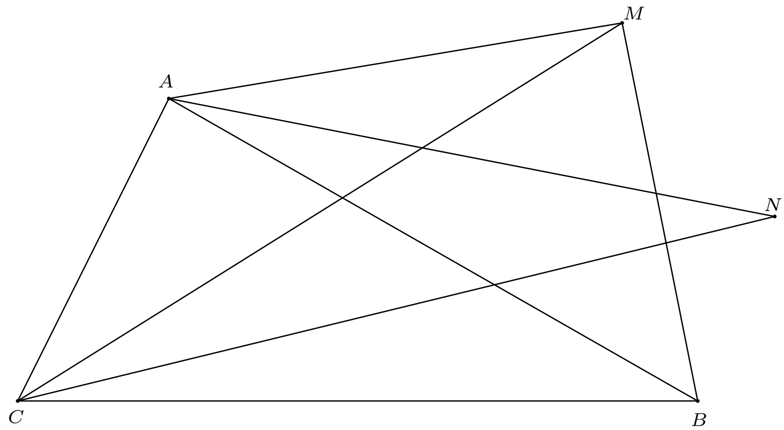

Figure 4: Fig. 4

Triangles ABM and CBM are *b.s.t.* Therefore by lemma 3.1. the triangle ACN is not *b.s.t*, which is contradiction.

Lemma 3.6. Let (A, B) and (C, D) be opposite sides pairs of a *g.s.q.* of a polygon. Then they could not have *g.p.v.* in one section.

Proof. Let in contrary, the pairs (A, B) and (C, D) have *g.p.v.* in one section.

Claim 1. Let the BC-*side-sector* is that one. Then any *g.p.v.* of the pair (A, D) must be in this section, Then and these three *g.p.v.* are different. But, the proof of the lemma 3.4. shows that we can change two of them with one common, which is impossible.

Claim 2. Let the section is AB-side-section or *CD-side section.* Then all pairs of the *g.s.q.* will have *g.p.v* in this section, which is impossible by Lemmas 3.2.

Now we are ready to do the proof of the **Theorem 2.3.**

Let in contrary, there is a n-gon with *g.s.q.* $ABCD$. It is clear that $n > 4$.

Claim 1. $n \leq 6$. By lemma 3.1. ABCD can not have more than one *b.s.t* of the n-gon

Out of ABCD there are not more than two *g.p.v.* Let A be a vertex such that any, subtriangle of $ABCD$ with vertex A is not *b.s.t.* Then two of pairs. $(A, B), (A, C)$ and (A, D) will have common *g.p.v.* which is impossible by lemma 3.5.

Claim 2. $n = 7$. Let $ABCD$ has a *b.s.t*, for example ABC. Then pairs $(D, A), (D, B)$ and (D, C) must have three different *g.p.v.* By lemma 3.3. the diagonal pair (D, B) has its *g.p.v.* in wide section which is one of two pairs-*side-section*: AB or CB. Without losing generality we can suppose that the pair (D,B) has *g.p.v.* in CB-*side-section.* Then the pair (D,A) have to have its *g.p.v* in this section, which is impossible by lemma 2.

Therefore ABCD does not have b.s.t.

D.7. We will say MN and XY are *same colour* if the good pairs (M, N) and (X, Y) have common *g.p.v.*

Now we have only three *g.p.v.* for four sides *good pairs.* Therefore two of them have -*same colour.* But it is impossible by lemmas 3.5. and 3.6.

Claim 3. $n = 8$. We have four *g.p.v.*

Therefore ABCD does not has *b.s.t.* and the four sides of ABCD must hare *different colours.* Then diagonal have to have same colour with one of them, with is impossible by lemm 3.5.

Claim 4. $n = 9$. The six *g.p.* of ABCD will have 5 different *g.p.v.* Four sides have different *colour*. Therefore two diagonals have *same colour*, ie. they have common *g.p.v.* suppose, in AB-side-section. Then pain (C, D) have to have its *g.p.v.* in this section. this is impossible by lemma 3.2. too

Claim 5.n=10. The six *g.p* of $ABCD$ have different 6 *g.p.v.* and dioganal pairs (A, C) and (B,D) hav *g.p.v.* in wide sections by lemma 3.3. Let BC and DC-*side-sections* are wide. If (A, C) and (B, D) have *g.p.v.* in one section, for example, in DC-*side-section*, then (A, B) have to have vibes *g.p.v.* in this section which is impossible by lemma 3.2.

If (A, C) and (B, D) have *g.p.v.* in different section, for example (A, C) has in DC-*side-section*, (B, D) has in BC-*side-section*, then *g.p.v* of (A,B) must be in DC-*side-section*, this is impossible by lemma 3.2. too.

Claim n $>$ 10 . Then by elimination of vertices of the n-gon which is not belongs in A, B, C, D and different from *g.p.v.* of the pairs of vertices of $ABCD$, we will have k-gon, with $k \leq 10$, in which ABCD is *g.s.q.* This is impossible by prevous claims. The proof of the Theorem 2.3. is completed.

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