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Simulation on Sangaku problem using optimization methods

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Abstract: Sangaku problem is one of Japanese Temple Geometry problems which was studied in Hidetoshi Fukugawa[1]. One of the Sangaku problem is packing 6 equal circles in rectangle of 1:1.934798 size. We examine the problem from a view point of optimization theory and algorithm. We show that Sangaku optimization problem belongs to a class of nonconvex optimization and propose a penalty method for solving the problem numerically. In numerical expirements, we consider equal and unequal 6 circles. Computational results obtained on Python Jupyter Notebook are provided.

Key words: Sangaku problem, packing problem, circle, local optimization

1. Introduction

The Sangaku problem is an ancient Japanese geometric problem founded in the 17 th-century in the Edo period (1603-1867). Sangaku means calculation tablet, were painted in color on wooden tablets and hung in the precincts of Buddhist temples and Shinto shrines all over Japan [1], as offerings to the kami and buddhas, as challenges to the congregants, or as displays of the solutions to questions. In the Edo period of Japan, there were neither universities nor colleges, so mathematicians had to make efforts to introduce mathematics to ordinary people on their own [2]. The oldest surviving sangaku tablet was recorded in 1683, about 900 tablets have survived, as well as several collections of sangaku problems in early 19th-century hand-written books or books produced from wooden blocks. The most notable peoples in Sangaku problems are Fujita Sadasuke(1734-1807), his son Fujita Kagen(1772-1828), and Ajima Naonobu(1732-1798) known as Ajima-Malfatti Points [3].

In this paper, we consider one of the Sangaku tablet problems with 6 equal circles in a rectangle on a rock from a Japanese temple (Figure 1). This Sangaku problem has been mentioned in paper [4] in 2020. This problem is a kind of circle packing problem such as packing circles in triangles, squares, and other geometrical figures [5-8].

This 6 circles problem was solved by Hidetoshi Fukagawa using a geometry method and obtained the length of the sides of this rectangle as 1×1.934798 (Figure 2).
We introduce an optimization approach to Sangaku 6 circle problem.

Figure 1: (Sangaku problem)

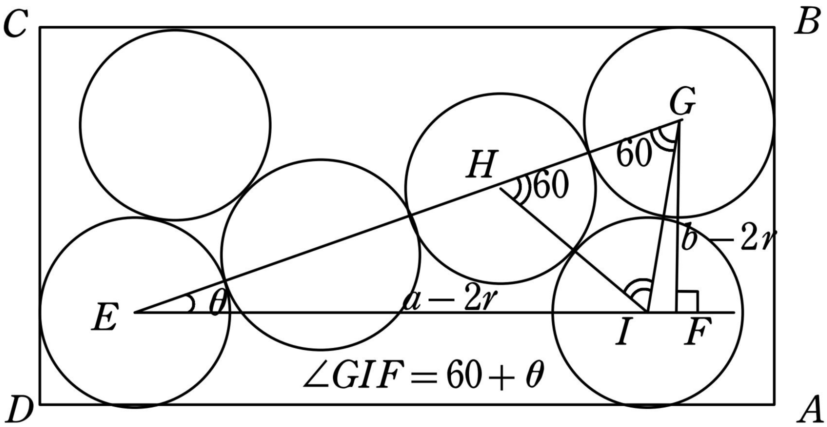

Figure 2: (Sangaku 6 circle problem)

It is a nonconvex optimization problem and belongs to a class of global optimization. Also, this optimization packing problem is extended for the case of unequal 6 tangent circles.To solve the problem locally, we use the Penalty function methods. Numerical experiments were performed by Python Jupyter Notebook.

2. Methodology

2.1. General formulation of packing problem

Define a sphere with radius r and center $u^0 \in \mathbb{R}^n$.

$$B(u^0, r) = \{x \in \mathbb{R}^n | \|u^0 - x\| \leq r\}$$

and let $D \subset \mathbb{R}^n$, here is D a bounded and polyhedral nonempty set

$$D = \{x \in \mathbb{R}^n | \langle a^i, x \rangle \leq b_i, a^i \in \mathbb{R}^n, b_i \in \mathbb{R}, \quad i = \overline{1, m}\}.$$

Theorem 2.1. [6] $B(u^0, r) \subset D$ *if and only if*

$$\langle a^i, x \rangle + r\|a^i\| \leq b_i, i = \overline{1, m} \tag{2.1}$$

The proof of theorem 2.1 is shown in paper [6].

Denote by $c^1, c^2, \ldots, c^k$ are centers of the spheres. Let $r_1, r_1, \ldots, r_k$ be their correspondind radii, respectively.

Optimization problem maximizing total volume of k non-overlapping spheres[9] is:

$$max_{(c,r)}\ V = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\sum_{i=1}^{k} r_i^n \tag{2.2}$$

subject to:

$$\langle a^i, c^j\rangle + r_j\|a^i\| \le b_i,\ \ i=\overline{1,m},\ \ j=\overline{1,k}. \tag{2.3}$$

$$\|c^i - c^j\|^2 \ge (r_i + r_j)^2,\ \ i,j=\overline{1,k},\ \ i<j. \tag{2.4}$$

$$r_i \ge 0,\ \ i=\overline{1,k} \tag{2.5}$$

here $\mathbb{D} = \{x \in \mathbb{R}^n | \langle a^i, x\rangle \le b_i\}$ is polytope, $a^i \in \mathbb{R}^n, b_i \in \mathbb{R},\ \ i=\overline{1,m}$. Γ is Euler's gamma function.

If $r_i = r,\ \ i=\overline{1,k}$ then problem (2.2)-(2.5) has the following form:

$$max_{(c,r)}\ V = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} k r^n \tag{2.6}$$

subject to:

$$\langle a^i, c^j\rangle + r\|a^i\| \le b_i,\ \ i=\overline{1,m},\ \ j=\overline{1,k}, \tag{2.7}$$

$$\|c^i - c^j\|^2 \ge 4r^2,\ \ i,j=\overline{1,k},\ \ i<j, \tag{2.8}$$

$$r \ge 0. \tag{2.9}$$

If we take $n=2,\ \ m=4,\ \ k=6$, it becomes Sangaku problem and $\mathbb{D}$ is:

$$D = \{(x_i, y_j) \in \mathbb{R}^2 \mid 0 \le x_i \le 1.934798,\ \ 0 \le y_j \le 1,\ \ \ i,j=\overline{1,6}\}.$$

2.2. Optimization for Sangaku problem

This Sangaku optimization problem for size of the rectangle 1×1.934798 with maximum density is formulated as follows:

$$f = \frac{6\pi}{1 \times 1.934798} r^2 \to \max \tag{2.10}$$

$$0 \le x_i \le 1.934798 - r,\ \ \ i=\overline{1,6} \tag{2.11}$$

$$0 \le y_i \le 1 - r,\ \ \ i=\overline{1,6} \tag{2.12}$$

$$\|c^i - c^j\|^2 \ge 4r^2;\ \ \ i<j;\ \ \ i,j=\overline{1,6} \tag{2.13}$$

$$r \ge 0 \tag{2.14}$$

where, $c^1(x_1,y_1), c^2(x_2,y_2), c^3(x_3,y_3), c^4(x_4,y_4), c^5(x_5,y_5), c^6(x_6,y_6)$ are the centers of six circles in a box set D and r is radius of circles.

Problem (2.10)-(2.14) has 13 variables and 45 nonlinear constraints

2.3. Sangaku Optimization problem for different radii

Now we consider Sangaku 6 circles optimization problem for the different radii.

$$f = \frac{\pi}{1 \times 1.934798}\sum_{i=1}^{6} r_i^2 \to \max \tag{2.15}$$

$$0 \le x_i \le 1.934798 - r_i;\ \ \ i=\overline{1,6} \tag{2.16}$$

$$0 \le y_i \le 1 - r_i;\ \ \ i=\overline{1,6} \tag{2.17}$$

$$\|c^i - c^j\|^2 \ge \|r_i + r_j\|^2;\ \ \ i<j;\ \ \ i,j=\overline{1,6} \tag{2.18}$$

$$r_i \ge 0;\ \ \ i=\overline{1,6} \tag{2.19}$$

where, $c^1(x_1,y_1), c^2(x_2,y_2), c^3(x_3,y_3), c^4(x_4,y_4), c^5(x_5,y_5), c^6(x_6,y_6)$ are the centers of six circles in a box set D and $r_1, r_2, \ldots, r_6$ are radii of circles.

Problem (2.15)-(2.19) has 18 variables and 45 nonlinear constraints.

2.4. Penalty function approach

For solving the problem locally , we use the penalty function method. Consider the following problem (2.20)-(2.22):

$$\min f(x), \quad x \in X \tag{2.20}$$

$$s.t. \quad g_i(x) \le 0 \quad i = \overline{1,m} \tag{2.21}$$

$$h_j(x) = 0 \quad j = \overline{1,s} \tag{2.22}$$

The penalty function method is expressed as follows[10]:

$$p_1(x) = \sum_{i=1}^{m} (\max\left(g_i(x), 0\right))^2 \tag{2.23}$$

$$p_2(x) = \sum_{j=1}^{s} h_j^2(x) \tag{2.24}$$

and

$$L(x, \rho_k) = f(x) + \rho_k(p_1(x) + p_2(x)) \tag{2.25}$$

then

$$x^k : L(x, \rho_k) \to \min \tag{2.26}$$

$$x \in X \tag{2.27}$$

Here $\{\rho_k\}_{k=1}^{\infty}$−is a positive sequence and it grows monotonously. We can replace minimum to maximum in problem (2.26)-(2.27), the following formulation

$$\max\{L\} = -\min\{-L\}. \tag{2.28}$$

The algorithms of Penalty function approach consists the following steps:

Algorithm: Penalty method

Step 1: Parameters ρ_1, m, eps are given and $k = 1, \rho_1 = 10$, $m \in N$

Step 2: Solve the minimization problem

$$\min_{x \in X} L(x, \rho_k) = \min_{x \in X} \left(f(x) + \rho_k(p_1(x) + p_2(x))\right). \tag{2.29}$$

get a local solution $x^k = \arg\min L(x, \rho_k), x \in X$.

Step 3: $u = x^k, \quad k = k + 1, \quad \rho_k = m\rho_{k-1}$

Step 4: Solve the following minimization problem

$$\min_{x \in X} L(x, \rho_k) = \min_{x \in X} \left(f(x) + \rho_k(p_1(x) + p_2(x))\right). \tag{2.30}$$

obtain another a local solution $x^k = \arg\min L(x, \rho_k), x \in X$.

Step 5: If $\|u - x^k\| \le eps$ then algorithm stops and solution x^k is a local minimum or a stationary point. Otherwise goto Step 6.

Step 6: $u = x^k$ and goto Step 3.

Convergence of the penalty function approach is given by the following theorem.

Theorem 2.2. [11] *Suppose f, g, h, p are continuous functions of problems (2.20)-(2.22). Let* $x^k = \arg\min f(x) + \rho_k p(x), \ x \in X$ *for a penalty function* $p(x)$ *as defined in subsection (2.23)-(2.24).*
Let $0 < \rho_1 < \rho_2 < \ldots < \rho_k$. *Let* $\hat{x}$ *be an arbitrary limit point of* $\{x^k\}_{k=1}^{\infty}$. *Then,* $\hat{x}$ *solves problems(2.20)-(2.22).*

3. The results

Problems (2.10)-(2.14) and (2.15)-(2.19) are solved numerically using algorithms of penalty method in Python Jupyter Notebook starting with several initial points.For problem (2.10)-(2.14), we found 3 local solutions and one of them coincided with the original Sangaku configuration (Figure 1), giving density 0.6137. Case 3 gives the best solution for equal 6 circles with 0.7075 density and 0.2694 radius.

Similarly, we found 6 solutions Problem (2.15)-(2.19) for the case of 6 unequal circles. Case 8 provides the best density 0.84.

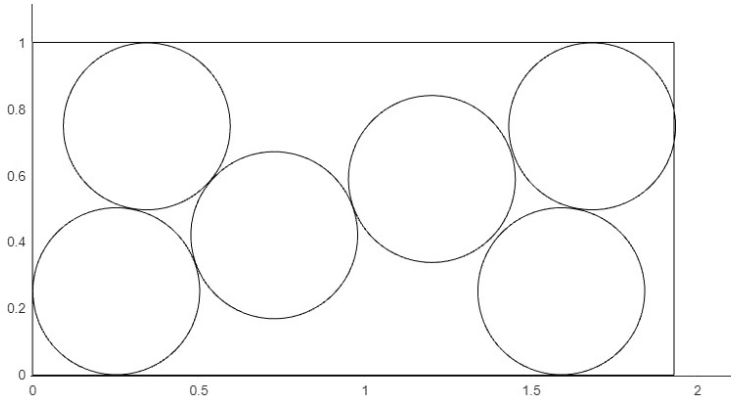

Figure 3: Case 1(Sangaku 6 equal circle problem)

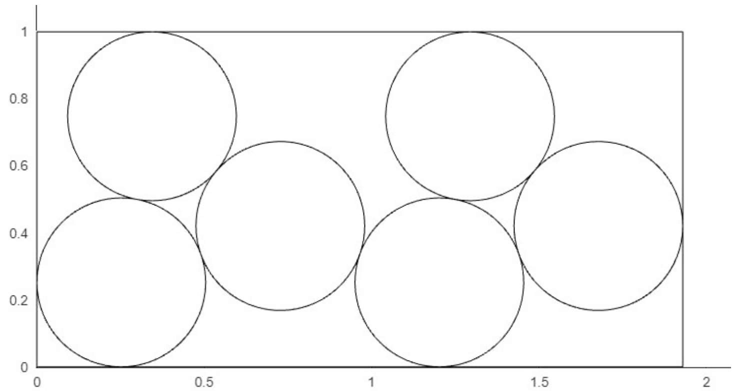

Figure 4: Case 2(Sangaku 6 equal circle problem)

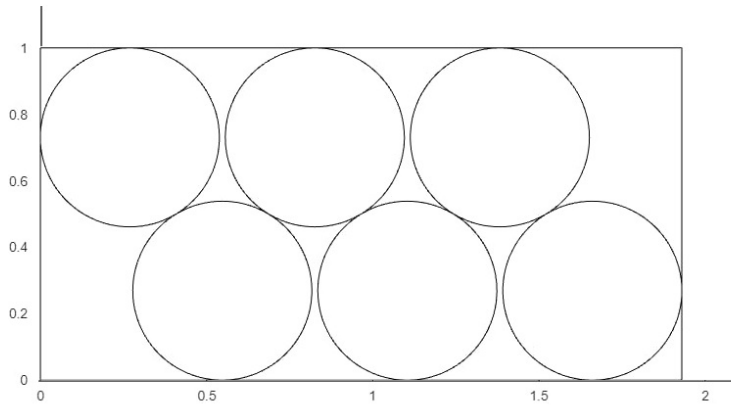

Figure 5: Case 3(Sangaku 6 equal circle problem)

Table 1: Numerical results of Case 1

$density : f$	**0.613**	
$radius : r$	**0.251**	
(x_1, y_1)	0.25	0.25
(x_2, y_2)	0.34	0.75
(x_3, y_3)	0.72	0.42
(x_4, y_4)	1.2	0.59
(x_5, y_5)	1.59	0.25
(x_6, y_6)	1.68	0.75

Table 2: Numerical results of Case 2

$density : f$	**0.619**	
$radius : r$	**0.2521**	
(x_1, y_1)	0.25	0.25
(x_2, y_2)	0.34	0.75
(x_3, y_3)	0.72	0.42
(x_4, y_4)	1.2	0.25
(x_5, y_5)	1.29	0.75
(x_6, y_6)	1.67	0.42

Table 3: Numerical results of Case 3

$density : f$	**0.7075**	
$radius : r$	**0.2694**	
(x_1, y_1)	0.27	0.73
(x_2, y_2)	0.55	0.29
(x_3, y_3)	0.83	0.73
(x_4, y_4)	1.1	0.27
(x_5, y_5)	1.38	0.73
(x_6, y_6)	1.66	0.29

Table 4: Numerical results of Case 4

$density : f$		**0.819**	
(x_1, y_1, r_1)	0.5	0.5	0.5
(x_2, y_2, r_2)	0.99	0.88	0.12
(x_3, y_3, r_3)	0.99	0.72	0.04
(x_4, y_4, r_4)	1	0.31	0.04
(x_5, y_5, r_5)	1.01	0.13	0.13
(x_6, y_6, r_6)	1.47	0.53	0.47

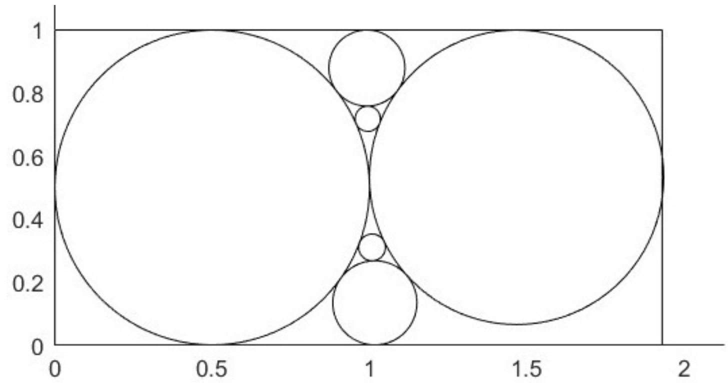

Figure 6: Case 4(Sangaku 6 different circle problem)

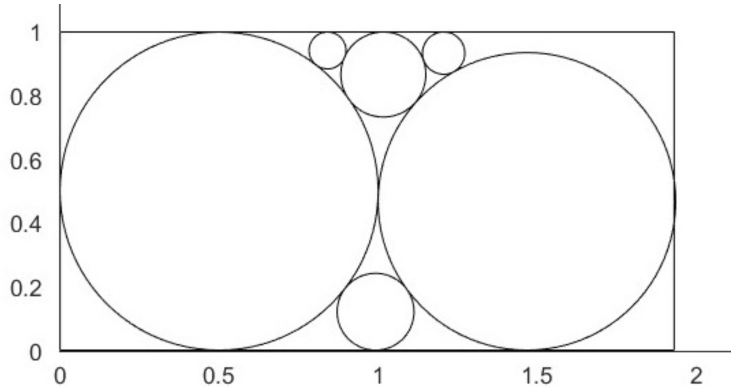

Figure 7: Case 5(Sangaku 6 different circle problem)

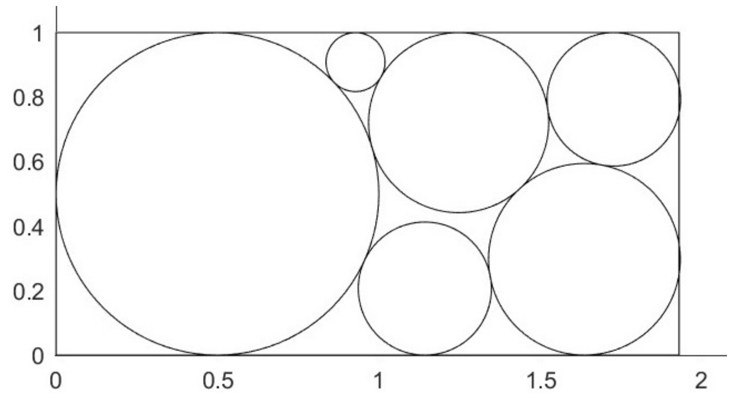

Figure 8: Case 6(Sangaku 6 different circle problem)

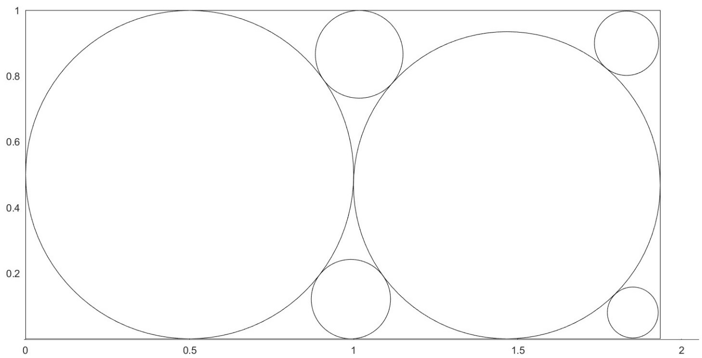

Figure 9: Case 7 (Sangaku 6 different circle problem)

Table 5: Numerical results of Case 5

$density : f$		**0.826**	
(x_1, y_1, r_1)	0.5	0.5	0.5
(x_2, y_2, r_2)	0.84	0.94	0.06
(x_3, y_3, r_3)	0.99	0.12	0.12
(x_4, y_4, r_4)	1.01	0.86	0.13
(x_5, y_5, r_5)	1.2	0.93	0.06
(x_6, y_6, r_6)	1.47	0.47	0.47

Table 6: Numerical results of Case 6

$density : f$		**0.828**	
(x_1, y_1, r_1)	0.5	0.5	0.5
(x_2, y_2, r_2)	0.93	0.9	0.09
(x_3, y_3, r_3)	1.14	0.2	0.2
(x_4, y_4, r_4)	1.24	0.72	0.28
(x_5, y_5, r_5)	1.64	0.3	0.3
(x_6, y_6, r_6)	1.73	0.73	0.2

Table 7: Numerical results of Case 7

$density : f$		**0.8391**	
(x_1, y_1, r_1)	0.5	0.5	0.5
(x_2, y_2, r_2)	1.47	0.47	0.47
(x_3, y_3, r_3)	1.85	0.08	0.078
(x_4, y_4, r_4)	1.02	0.87	0.137
(x_5, y_5, r_5)	1.83	0.9	0.098
(x_6, y_6, r_6)	0.99	0.12	0.12

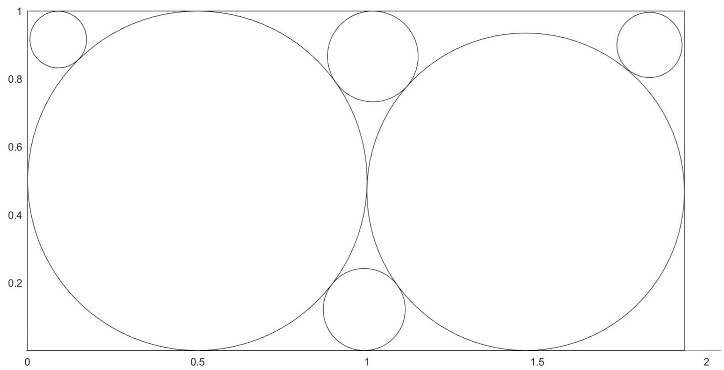

Figure 10: Case 8 (Sangaku 6 different circle problem)

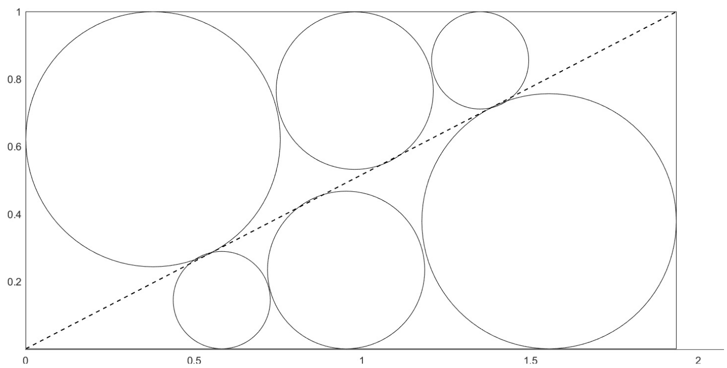

Figure 11: Case 9 (Sangaku 6 different circle problem)

Table 8: Numerical results of Case 8

$density : f$		**0.84**	
(x_1, y_1, r_1)	0.5	0.5	0.5
(x_2, y_2, r_2)	1.47	0.47	0.47
(x_3, y_3, r_3)	0.09	0.916	0.084
(x_4, y_4, r_4)	1.02	0.87	0.137
(x_5, y_5, r_5)	1.83	0.9	0.096
(x_6, y_6, r_6)	0.99	0.12	0.12

Table 9: Numerical results of Case 9

$density : f$		**0.7102**	
(x_1, y_1, r_1)	0.37	0.62	0.37
(x_2, y_2, r_2)	1.35	0.85	0.14
(x_3, y_3, r_3)	0.97	0.76	0.23
(x_4, y_4, r_4)	1.55	0.37	0.37
(x_5, y_5, r_5)	0.58	0.14	0.14
(x_6, y_6, r_6)	0.95	0.23	0.23

4. Conclusions

We consider 6 circles packing problem from a view point of optimization theory and method.The new formulated optimization problem belongs to a class of global optimization using penalty method, we have made attempt to solve the problem locally.The numerical results were conducted on Python Jupyter notebook for equal and unequal circles in the rectangle of Sangaku size 1×1.934798.

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Сангаку бодлогыг оптимизацийн аргаар бодох нь

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Хураангуй: Сангаку бодлого нь Японы эртний геометрийн бодлого юм. Энэхүү судалгааандаа бид Хидетоши Фукугава эрдэмтний судалсан 1:1.934798 хэмжээтэй тэгш өнцөгтөд 6 ижил тойрог багтаах сангаку бодлогыг авч үзэв. Энэ бодлогыг хучилтын бодлогын хүрээнд оптимизацийн аргаар бодсон ба энэ нь гүдгэр бус максимумчлалын бодлого болно.Бодлогыг өргөтгөж, ижил бус 6 тойргын хувьд бодож торгуулийн функцийн аргаар нэмж тооцооллыг хийв. Python Jupyter Notebook программ дээр тооцооллыг хийж үр дүнг гаргасан болно.

Түлхүүр үгс: Сангаку бодлого, хучилтын бодлого, тойрог, оптимизацийн арга