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Malfatti's constrained optimization problem

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Abstract: In 1803 Italian mathematician Malfatti posed the following problem how to pack three non-overlapping circles of maximum total area in a given triangle. Malfatti originally assumed that the solution to this problem are three circles inscribed in a triangle such that each circle tangent to other two and touches two sides of the triangle. Now it is well known that Malfatti's solution is not optimal. The problem for the first time was treated as a global optimization problem in Enkhbat [9]. In this paper, we consider a new formulation of Malfatti's problem called Malfatti's constrained optimization problem. The new problem is formulated as a nonconvex optimization problem with nonlinear constraints. Numerical experiments were conducted on Python for the cases.

Key words: Malfatti's problem, Nonconvex optimization, circle, triangle

1. Introduction

Malfatti's Problem is a classical geometric optimization problem that deals with finding the arrangement of three non-overlapping circles with maximum total area inside a given triangle. Originally, it was assumed that the optimal solution would be three circles inscribed in the triangle in such a way that each circle is tangent to the other two and also touches two sides of the triangle. However, it was later discovered that this arrangement is not always optimal in terms of maximizing the total area of the circles.

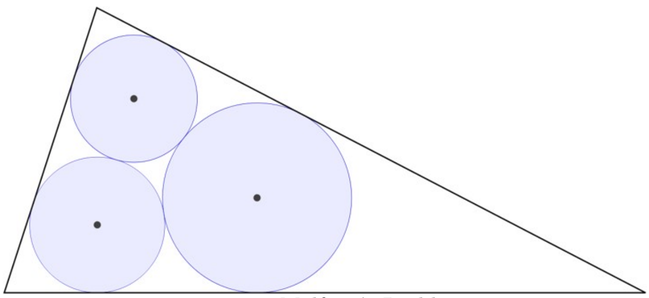

Figure 1: Malfatti's Problem

Over the years, various methods and approaches have been employed to find the best solutions to Malfatti's Problem [1–8]. These methods include algebraic and geometric approaches, as well as trigonometric equations and inequalities. In 1994, Zalgaller and Los [3] demonstrated that a greedy arrangement, where the circles are arranged with a different configuration, can yield a better solution. They used trigonometric equations and inequalities, along with the concept of "rigid systems," to find the optimal solution.

Non-convex optimization problems are generally more challenging to solve because they involve functions that may have multiple local optima, making it difficult to find the global optimum. In the context of Malfatti's Problem, this formulation might consider various geometric and trigonometric constraints on the circle arrangements within the triangle, which are not easily amenable to standard linear optimization methods. It's worth noting that the problem of finding the best arrangement of circles in a triangle was treated as a global optimization problem in Enkhbat [9].

Reference to "Malfatti's constrained optimization problem" suggests that there may be a new formulation of the problem that introduces non-convex optimization and nonlinear constraints, which can make it more challenging and interesting from a mathematical and computational perspective. Solving such a problem would likely involve advanced optimization techniques and algorithms to find the optimal circle arrangement that maximizes the total area while satisfying the specified constraints. This is an example of how classical geometric problems can be extended and reformulated to incorporate modern mathematical optimization techniques, opening up new avenues for research and exploration.

The goal in this formulation of Malfatti's Problem would be to find the optimal arrangement of non-overlapping circles within the given triangle, maximizing the total area of the circles, while adhering to the specified nonlinear constraints. It's a fascinating example of how mathematical optimization can be applied to solve real-world geometric problems with various complexities and constraints, and it may have applications in fields such as architecture, design, or operations research.

In this paper, we consider a new formulation of Malfatti's problem called Malfatti's constrained optimization problem. The problem is formulated as a nonconvex optimization problem with nonlinear constraints.

2. Methodology

2.1. Malfatti's problem and convex maximization

In the process of transforming Malfatti's problem into an optimization problem, several key steps must be undertaken. Initially, it is imperative to express the problem in equivalent terms, which involves the utilization of convex sets, notably a ball and a triangle set. Subsequently, the focus shifts towards establishing the conditions that govern the inscribed placement of balls within a triangle set. To facilitate this endeavor, a set of relevant sets is introduced, serving as foundational components for further analysis and problem formulation.

Let $B(x,r)$ be a ball with a center $x \in \mathbb{R}^n$ and radius $r \in \mathbb{R}$,

$$B(x,r) = \{y \in \mathbb{R}^n | \|y-x\| \le r\} \tag{2.1}$$

Assume that D is a compact set which is not congruent to a sphere and $intD \neq \emptyset$. Clearly, D is convex set in $\mathbb{R}^n$. Let D be a polyhedral set given by the following linear inequalities.

$$D = \{y \in \mathbb{R}^n | \langle a^i, y\rangle \le b_i,\ i=\overline{1,m}\}, \quad a^i \in \mathbb{R}^n, \quad b_i \in \mathbb{R}, \tag{2.2}$$

here $\langle , \rangle$ denotes the scalar product of two vectors in $\mathbb{R}^n$, and $\|\cdot\|$ is Euclidean norm, $a^i \neq a^j,\ i \neq j;\ i,j=\overline{1,m}$.

Theorem 2.1. *[9] $B(x,r) \subset D$ if and only if*

$$\langle a^i, x\rangle + r\|a^i\| \le b_i, \quad i=\overline{1,m}. \tag{2.3}$$

Now we formulate inscribed conditions of three balls into a triangle set.

Assume that intersections of interiors of these balls are empty. One of the balls is tangent to other two but their centers don't lie on the same line. At the same time, the anytwo balls don't intersect with each other.

Denote by $c^1(x_1, x_2), c^2(x_4, x_5)$ and $c^3(x_7, x_8)$ centers of three balls inscribed in a triangle set D given by (2). Let x_3, x_6 and x_9 be their corresponding radii. Malfatti's problem following:

$$max f = \pi(x_3^2 + x_6^2 + x_9^2), \quad (2.4)$$

$$\langle a^i, u\rangle + x_3\|a^i\| \leq b_i, \quad c^1 = (x_1, x_2), \quad i = 1, 2, 3, \quad (2.5)$$

$$\langle a^i, v\rangle + x_6\|a^i\| \leq b_i, \quad c^2 = (x_4, x_5), \quad i = 1, 2, 3, \quad (2.6)$$

$$\langle a^i, p\rangle + x_9\|a^i\| \leq b_i, \quad c^3 = (x_6, x_7), \quad i = 1, 2, 3, \quad (2.7)$$

$$(x_4 - x_1)^2 + (x_5 - x_2)^2 - (x_3 + x_6)^2 \geq 0, \quad (2.8)$$

$$(x_7 - x_1)^2 + (x_8 - x_2)^2 - (x_3 + x_9)^2 \geq 0, \quad (2.9)$$

$$(x_7 - x_4)^2 + (x_8 - x_5)^2 - (x_6 + x_9)^2 \geq 0, \quad (2.10)$$

$$x_3 \geq 0, \quad x_6 \geq 0, \quad x_9 \geq 0. \quad (2.11)$$

The function f in (2.4) denotes a total area of the three balls. Thus, problem (2.4) – (2.11) becomes the convex maximization problem over a nonconvex set.

2.2. Malfatti's constrained optimization problem

In this section, we extend Malfatti's problem as follows. Let the centers of the circles included in the triangle lie on the given curves $g_1(c^1)$, $g_2(c^2)$, $g_3(c^3)$. Malfatti's constrained optimization problem is the following:

$$g_1\left(c^1\right) = 0, \quad g_2\left(c^2\right) = 0, \quad g_3\left(c^3\right) = 0. \quad (2.12)$$

$g_1(c^1), g_2(c^2), g_3(c^3)$ – any functions. (2.4)–(2.12) is called Malfatti's constrained optimization problem.

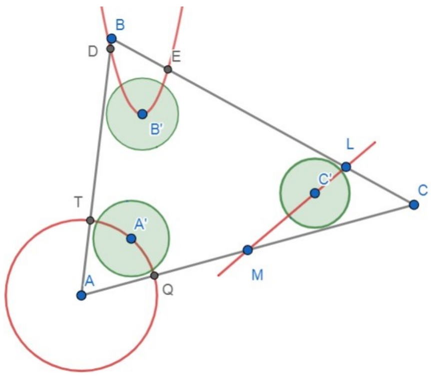

Figure 2: Example: Malfatti's constrained optimization problem

2.3. The Gekko Optimization Algorithm

The Gekko optimization algorithm is a powerful numerical optimization tool designed for solving complex optimization problems across various domains, including engineering, finance, and operations research. Developed to address both constrained and unconstrained optimization challenges, Gekko leverages a combination of optimization techniques to efficiently find optimal solutions.

We used APOPT solver for Gekko. APOPT is a nonlinear programming solver used in the Gekko optimization modeling language. Gekko is a Python library for optimization and model predictive control. APOPT, or A Large-scale Approximate Networked Interior Point Optimizer, is an optimization solver that focuses on large-scale nonlinear programming problems. It's particularly well-suited for solving mixed-integer and mixed-integer nonlinear programming problems. The mathematics behind APOPT involves interior point optimization techniques. Interior point methods are a class of algorithms used for solving nonlinear programming problems. These methods iteratively move towards the optimal solution by exploring the interior of the feasible region, as opposed to moving along the boundary. The interior point optimization method:

Barrier Function: Interior point methods use a barrier function to penalize infeasible points. This function increases as the solution approaches the boundary of the feasible region, effectively guiding the optimization process toward the interior.

Central Path: The algorithm follows a central path, which is a trajectory in the feasible region where the barrier function is minimized. The central path helps maintain feasibility while moving towards the optimal solution.

Optimality Conditions: The algorithm seeks points that satisfy both the equality and inequality constraints of the optimization problem. At the optimal solution, the first-order optimality conditions (Karush-Kuhn-Tucker conditions) are satisfied.

Iterative Refinement: The algorithm iteratively refines the solution, adjusting the penalty parameter and updating the search direction in each iteration until convergence is achieved.

3. The results

In computational experiments we used Gekko which is a Python library for optimization and modeling of dynamic systems. It is commonly used for solving mathematical optimization problems, particularly those involving differential equations and dynamic simulations. For a test purpose, the triangle with vertices $A(-4,-4), B(2,6)$ and $C(6,-2)$ has been considered. In this calculation, the initial approximation points are randomly taken, and the local solutions are found. Now we have the following cases:

Case 1: *$g_1(c^1), g_2(c^2), g_3(c^3)$ – be the medians drawn from the three vertices of triangle ABC.*

$$
\begin{aligned}
max f &= \pi(x_3^2 + x_6^2 + x_9^2),\\
-10x_1 + 6x_2 + x_3\sqrt{136} &\le 16,\\
x_1 - 5x_2 + x_3\sqrt{26} &\le 16,\\
2x_1 + x_2 + x_3\sqrt{5} &\le 10,\\
-10x_4 + 6x_5 + x_6\sqrt{136} &\le 16,\\
x_4 - 5x_5 + x_6\sqrt{26} &\le 16,\\
2x_4 + x_5 + x_6\sqrt{5} &\le 10,\\
-10x_7 + 6x_8 + x_9\sqrt{136} &\le 16,\\
x_7 - 5x_8 + x_9\sqrt{26} &\le 16,\\
2x_7 + x_8 + x_9\sqrt{5} &\le 10,\\
(x_4 - x_1)^2 + (x_5 - x_2)^2 - (x_3 + x_6)^2 &\ge 0,\\
(x_7 - x_1)^2 + (x_8 - x_2)^2 - (x_3 + x_9)^2 &\ge 0,\\
(x_7 - x_4)^2 + (x_8 - x_5)^2 - (x_6 + x_9)^2 &\ge 0,\\
g_1(c^1) = 6x_1 - 8x_2 - 8,\ g_2(c^2) = 3x_4 + 7x_5 - 4,\ g_3(c^3) &= 9x_7 - x_8 - 12,\\
x_3 \ge 0,\ x_6 \ge 0,\ x_9 &\ge 0.
\end{aligned}
$$

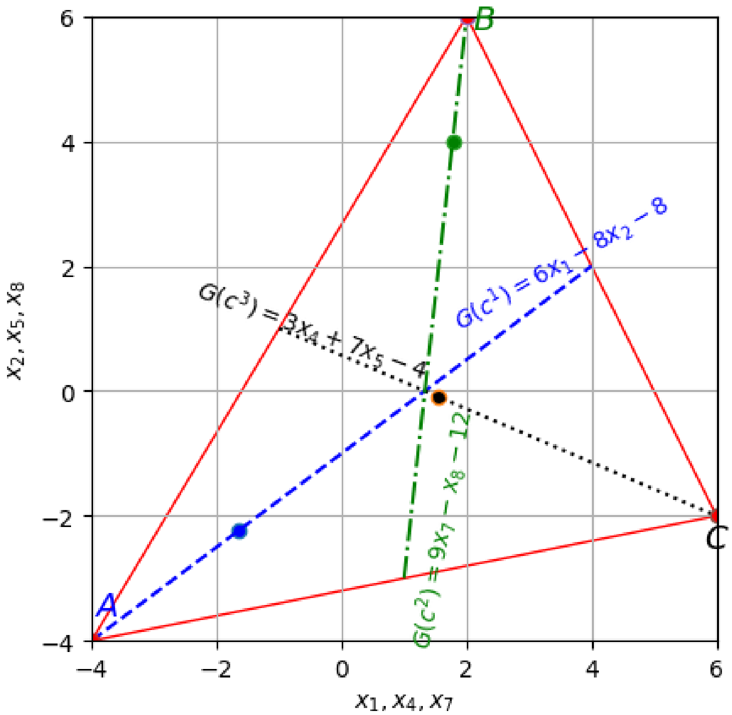

Figure 3: Case 1

Case 2: *$g_1(c^1)$ – is the line passing through sides AB and AC, $g_2(c^2)$ – circle centered on the vertex C, $g_3(c^1)$ – parabola drawn close to the vertex B of ABC triangle.*

$$
\begin{aligned}
max f &= \pi(x_3^2 + x_6^2 + x_9^2),\\
-10x_1 + 6x_2 + x_3\sqrt{136} &\leq 16,\\
x_1 - 5x_2 + x_3\sqrt{26} &\leq 16,\\
2x_1 + x_2 + x_3\sqrt{5} &\leq 10,\\
-10x_4 + 6x_5 + x_6\sqrt{136} &\leq 16,\\
x_4 - 5x_5 + x_6\sqrt{26} &\leq 16,\\
2x_4 + x_5 + x_6\sqrt{5} &\leq 10,\\
-10x_7 + 6x_8 + x_9\sqrt{136} &\leq 16,\\
x_7 - 5x_8 + x_9\sqrt{26} &\leq 16,\\
2x_7 + x_8 + x_9\sqrt{5} &\leq 10,\\
(x_4 - x_1)^2 + (x_5 - x_2)^2 - (x_3 + x_6)^2 &\geq 0,\\
(x_7 - x_1)^2 + (x_8 - x_2)^2 - (x_3 + x_9)^2 &\geq 0,\\
(x_7 - x_4)^2 + (x_8 - x_5)^2 - (x_6 + x_9)^2 &\geq 0,\\
g_1(c^1) = 2x_1 + x_2 + 4,\ g_2(c^2) = (x_4 - 6)^2 + (x_5 + 2)^2 - 16,\ g_3(c^3) &= x_7^2 - x_8 + 7,\\
x_3 \geq 0,\ x_6 \geq 0,\ x_9 &\geq 0.
\end{aligned}
$$

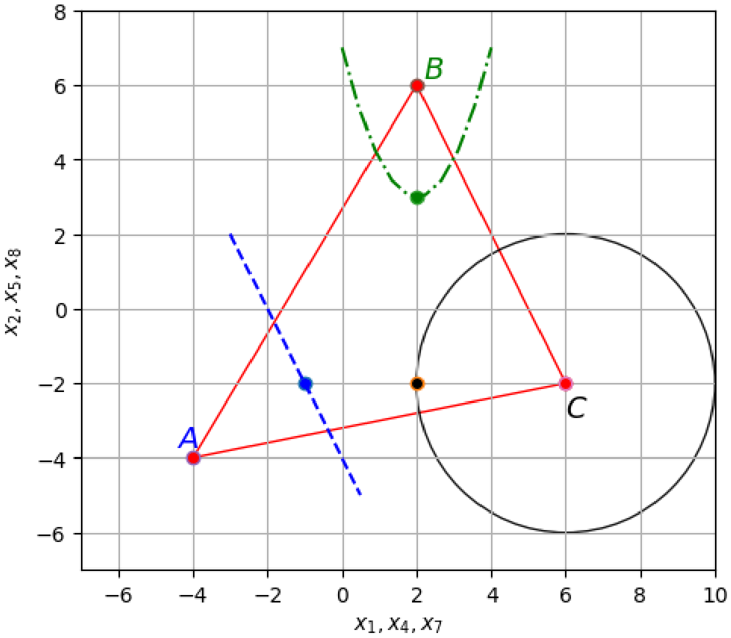

Figure 4: Case 2

The performance of the proposed algorithm was tested on the above constrained Malfatti's problems. The programming code for the algorithm was written in Colab of python and run online. The results are given for in Figure 5, 6.

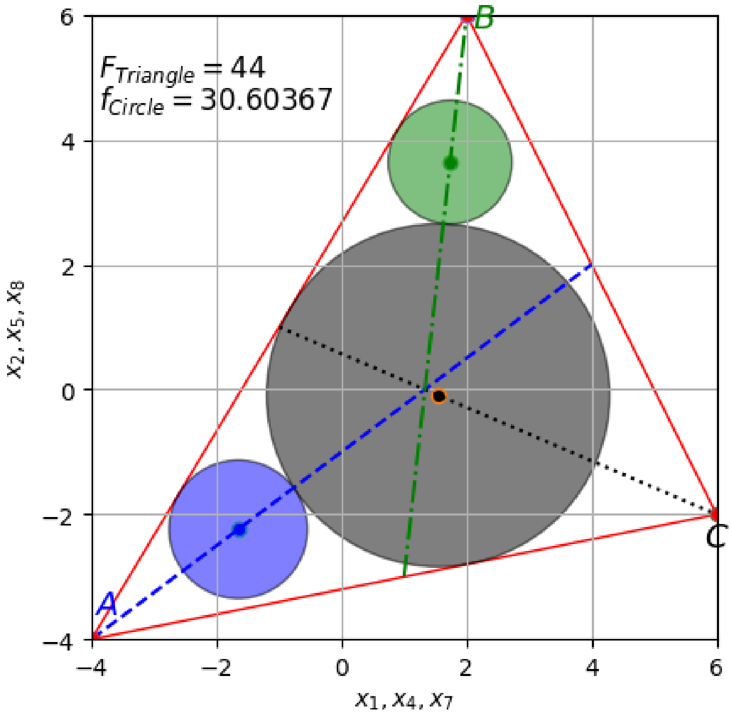

Figure 5: Case 1 solution

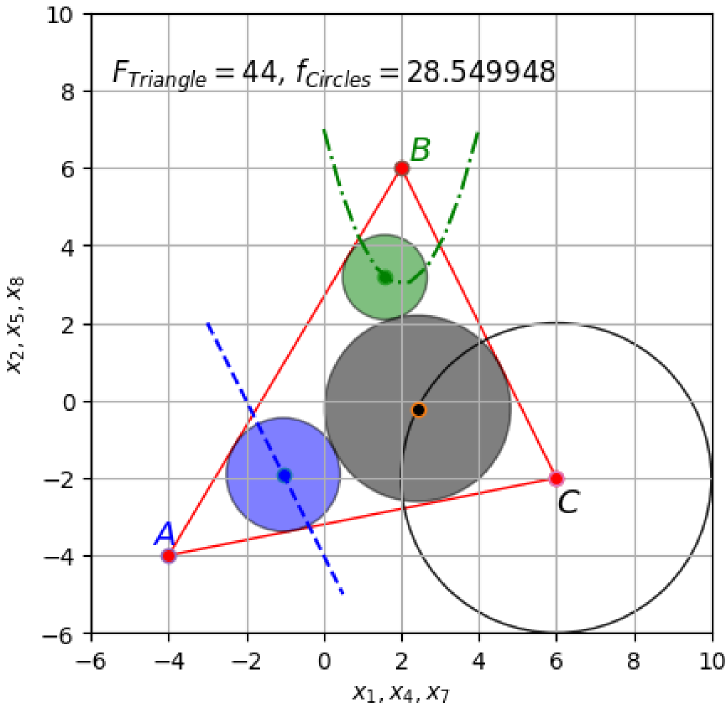

Figure 6: Case 2 solution

The figure 5 shows that the optimal solution for case 1 is the largest circle with the center located on the obtuse side of the triangle, while the smallest circle is located on the acute side. We have tested two cases of Malfatti's problem. The results are given for each case in Table 1.

Table 1: Ruselt

Case	Ratio/Percentage of packed area	Iteration	Computational time(s)
Case 1	69.553%	8	0,0115сек
Case 2	64.886%	9	0,0391сек

4. Conclusions

The 200 years old Malfatti's problem was extended and viewed from a view point of the convex maximization problem. We formulate a now global optimization problem based on Malfatti's problem by introducing constraints on the centers of Malfatti's circles. In order to solve the problem locally, we use Gekko package in python. Numerical experiments were done for two cases of constraints. 69% and 64% of the total area were packed by 3 different the circles. The case 1 corresponds to the largest circle of the 3 circles located on the obtuse side of the triangle. The problem can be further extended for the high dimensional cases with more than 3 circles. It will be considered in a next paper.

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Мальфаттын Зааглалтай Бодлого

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Өгүүллийн мэдээлэл: Хүлээн авсан: 2023.09.15; Зөвшөөрөгдсөн: 2023.10.01; Нийтлэгдсэн: 2023.12.26

Хураангуй: 1803 онд анх Италийн математикч Мальфатт өгөгдсөн гурвалжинд хамгийн их талбайтай, давхцахгүйгээр гурван тойргийг хэрхэн байрлуулах вэ? гэсэн бодлогыг тавьж байсан бөгөөд энэхүү бодлогын шийд нь гурвалжинд багтсан гурван тойргуудын тойрог бүр нөгөө хоёр тойргийг, гурвалжны хоёр талыг шүргэсэн байна гэж үзсэн. Энэ нь хараахан оновчтой шийд биш байсан ба [9] ажилд анх Мальфаттын бодлогыг бодох глобал оновчлолын бодлогыг томьёолж шийдийг олох арга алгоритм боловсруулсан. Энэхүү судалгаанд бид Мальфаттын бодлогын шинэ томьёолол буюу Хөдөлгөөнт Мальфаттын бодлогыг авч үзсэн. Тус бодлого нь шугаман бус хязгаарлалттай гүдгэр бус оновчлолын бодлого юм. Тоон туршилтыг Python дээр хийсэн.

Түлхүүр үгс: Мальфаттын бодлого, гүдгэр бус оновчлол, тойрог, гурвалжин