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Existence of blowing-up solutions to some Schrödinger equations including nonlinear amplification with small initial data

Naoyasu Kita

Faculty of Advanced Science & Technology, Kumamoto University, Japan

ABSTRACT. We consider the existence of blowing-up solutions to some Schrödinger equations including nonlinear amplification. The blow-up is considered in $L^2(\mathbb{R})$. Even though initial data are taken so small, there exist some solutions blowing-up in finite time. The theorem in this paper is an extension of Cazenave-Martel-Zhao's result [7] from the point of making the lower bound of power of nonlinearity extended and from the point of ensuring that blowing-up solutions exist even for small initial data.

KEYWORDS. Nonlinear Schroedinger equation, nonlinear amplification, blowing-up solution, small initial data.

1. INTRODUCTION

We consider the Cauchy problem of a nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u = -\frac{1}{2}\partial_x^2 u + (\lambda + i\kappa)|u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where the complex-valued unknown function $u = u(t,x)$ is defined on $(t,x) \in [0,T) \times \mathbb{R}^1$. In the nonlinearity, the power satisfies $2 < p \leq 3$ and the coefficients $\lambda, \kappa \in \mathbb{R}$ satisfy

$$\kappa > 0, \quad (p-1)|\lambda| \leq 2\sqrt{p}\,\kappa. \tag{2}$$

In particular, the positivity of κ in (2) implies that the nonlinearity affects as an amplification. To see it, we refer to the idea of Zhang [16]. If the region of x is a bounded interval I and Dirichlet boundary condition is imposed, then it is easy to show that, for $u_0 \in L^2(\mathbb{R})$ and $u_0 \neq 0$, the solution to (1) blows up in finite time. In fact, we have

$$\begin{aligned} \frac{d\|u(t)\|_{L^2(I)}^2}{dt} &= 2\mathrm{Re}(u(t), \partial_t u(t))_{L^2(I)} \\ &= 2\kappa\|u(t)\|_{L^{p+1}(I)}^{p+1}, \end{aligned}$$

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[†] Corresponding author. nkita@kumamoto-u.ac.jp.

where $(f,g)_{L^2(I)} = \int_I f(x)\overline{g(x)}dx$ is the usual L^2-inner product. Applying Hölder's inequality : $|I|^{(p-1)/2}\|u(t)\|^{p+1}_{L^{p+1}(I)} \geq \|u(t)\|^{p+1}_{L^2(I)}$ where $|I|$ denotes the size of the interval, we see that

$$\frac{d\|u(t)\|^2_{L^2(I)}}{dt} \quad \geq \quad 2\kappa |I|^{-(p-1)/2}\|u(t)\|^{p+1}_{L^2(I)}.$$

Solving this differential inequality, we have

$$\|u(t)\|_{L^2(I)} \geq \frac{\|u_0\|_{L^2(I)}}{\left\{1-\kappa(p-1)|I|^{-(p-1)/2}\|u_0\|^{p-1}_{L^2(I)}t\right\}^{1/(p-1)}},$$

and we know that $\|u(t)\|_{L^2(I)}$ blows up in finite time. However, this kind of estimate holds only in the case that x belongs to the bounded interval. Once the region becomes unbounded, the dispersion associated with $-\frac{1}{2}\partial_x^2$ will work so that the nonlinear amplification is suppressed, and it is difficult to presume that the nonlinear amplification surely generates a blowing-up solution. Actually when $3 < p$ and u_0 is sufficiently small in $H^1(\mathbb{R})$ with $xu_0 \in L^2(\mathbb{R})$ also small, the solution to (1) exists globally in time. This is because $|u(t,x)|^{p-1} \sim Ct^{-(p-1)/2}$ is integrable for large t, and the nonlinearity does not affect to the behavior of the solution. This observation suggests that, if we expect the blow-up for a small initial data, it is necessary to assume $p \leq 3$.

Our goal is to obtain blowing-up solutions to (1) even though the smallness is assumed on the initial data.

Theorem 1.1. *Let $2 < p \leq 3$. Also let λ and κ satisfy (2). Then, for any $\rho > 0$, there exists some initial datum $u_0 \in L^2(\mathbb{R})$ such that*

(i) $\|u_0\|_{L^2(\mathbb{R})} < \rho$,

(ii) *the solution u to (1) with u_0 as the initial datum satisfies*

$$\lim_{t\uparrow T*} \|u(t)\|_{L^2(\mathbb{R})} = \infty \tag{3}$$

for some $T^* > 0$.

Theorem 1.1 asserts the existence of a blowing-up solution only for some special small initial data. It remains open whether any small initial data except for $u_0 = 0$ give rise to the blow-up. However, Cazenave-Correia-Dickstein-Weissler [1] proved that any nontrivial solution in $H^1(\mathbb{R})$ is estimated from below with a function of t growing-up to ∞ as $t \to \infty$ – the solution may either blow up in finite time or grow up at $t = \infty$.

In Theorem 1.1, the lower bound of p is required for the technical reason that the blowing-up profile must be integrable around the blowing-up time with respect to t. The upper bound of p is required to ensure the existence of blowing-up solution for small initial data. Precisely speaking, we will first construct a blowing-up profile, construct a solution to (1) which approaches to the profile while $t \uparrow T*$, and extend it backward in time. In order to guarantee the decay of the solution in the negative time-direction, the assumption of $p \leq 3$ is required.

The construction of a blowing-up solution to some Schrödinger equation with nonlinear source term was considered by Cazenave-Martel-Zhao [7]. They treated the N-dimensional nonlinear Scrödinger equation :

$$i\partial_t u = -\Delta u + i|u|^{p-1}u,$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $\Delta = \sum_{j=1}^N \partial_{x_j}^2$. In their idea, a profile $\varphi(t,x)$ of the blow-up solution was firstly determined, which is subject to the ODE :

$$i\partial_t \varphi(t,x) = i|\varphi|^{p-1}\varphi(t,x).$$

They employed, for instance, $\varphi(t,x) = ((p-1)|t| + A|x|^k)^{-1/(p-1)}$ for some $A, k > 0$, which blows up at $t = 0$, and solve the nonlinear Schrödinger equation in $H^1(\mathbb{R}^N)$ by setting $u(t,x) = \varphi(t,x) + v(t,x)$ with $v(0,x) = 0$. However, in [7], the blow up for "small" initial data was not considered. Also, in their argument, the condition $3 \leq p$ was assumed. We extend this restriction to $2 < p$ by somewhat sophisticated nonlinear estimate as well as the coefficient of nonlinearity is generalized as in (2). For another progress on the large-data-blow-up, we refer to [3, 10]. We will not consider N-dimensional problem since the p must be restricted into $p \leq 1 + 2/N$ and the $\varphi(t,x) = O(|t|^{-1/(p-1)})$ admits a non-integrable singularity if $N \geq 2$.

2. Blowing-Up Profiles

We expect that the blow-up of the solutions is caused by the nonlinearity, and so the dispersion associated with $-\frac{1}{2}\partial_x^2$ does not work so strongly just before the blowing-up time. This observation suggests that the blowing-up profile is subject to the ordinary differential equation :

$$i\partial_t \varphi(t,x) = (\lambda + i\kappa)|\varphi(t,x)|^{p-1}\varphi(t,x). \tag{4}$$

For (4), we impose an initial data $\varphi(-1,x) = \varphi_{-1}(x)$ at each $x \in \mathbb{R}$, where φ_{-1} satisfies

(A)*The assumption on* φ_{-1}:

- (A.1) The $\varphi_{-1} \in C_0^\infty(\mathbb{R})$ is real valued.
- (A.2) $0 \leq \varphi_{-1}(x) \leq (\kappa(p-1))^{-1/(p-1)}$.
- (A.3) $\varphi_{-1}(x) = (\kappa(p-1))^{-1/(p-1)}$ if and only if $x = 0$.
- (A.4) $\varphi_{-1}(x) = (\kappa(p-1))^{-1/(p-1)}(1 - x^{2N})^{1/(p-1)}$ for $|x| < 1/2$, where $N > 0$ is sufficiently large integer.
- (A.5) $\varphi_{-1}(x) \leq \varphi_{-1}(1/2)$ for $|x| \geq 1/2$.

The ODE in (4) is easy to slove. In fact, by (4), we see that

$$\partial_t |\varphi(t,x)|^2 = 2\kappa |\varphi(t,x)|^{p+1},$$

which yields

$$\partial_t |\varphi(t,x)|^{-(p-1)} = -\kappa(p-1). \tag{5}$$

Integrating (5) from -1 to $t < 0$, we have

$$|\varphi(t,x)| = \frac{|\varphi_{-1}(x)|}{\{1 - \kappa(p-1)|\varphi_{-1}(x)|^{p-1}(t+1)\}^{1/(p-1)}}. \tag{6}$$

Substitute (6) into the $|\varphi(t,x)|^{p-1}$ on the right hand side of (4). Then we notice that it is a standard first order ODE of $\varphi(t,x)$, and we obtain

$$\varphi(t,x) = \varphi_{-1}(x)\left\{1 - \kappa(p-1)\varphi_{-1}^{p-1}(x)(t+1)\right\}^{(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}}. \tag{7}$$

We call $\varphi(t,x)$ in (7) *the blowing-up profile*. By the assumption (A) on φ_{-1}, the $\varphi(t,x)$ blows up at $t = 0$, and, precisely speaking, $\lim_{t\uparrow 0} |\varphi(t,0)| = \infty$ occurs but $|\varphi(0,x)| < \infty$ for $x \neq 0$. The condition (A.4) suggests that the graph of $\varphi_{-1}(x)$ is quite flat around $x = 0$, which guarantees that the blowing-up rates of $\partial_x\varphi(t,x)$ and higher derivatives do not violate the integrability with respect to t around $t = 0$ when $2 < p$. We will see, without proof, the detail on $\varphi(t,x)$ in next lemma.

Lemma 2.1. *Let φ_{-1} be such as defined in the assumption* (A)*, and let j be an integer satisfying $0 \leq j \leq N$. Then there exists some $C_j > 0$ such that the blowing-up profile* (7) *satisfies*

$$|\partial_x^j \varphi(t,x)| \leq C_j |t|^{-1/(p-1)-j/(2N)} \tag{8}$$

for any $t \in (-1,0)$.

3. A Solution Around the Blowing-Up Profile

We will construct a solution to (1) locally in negative time, which asymptotically tends to $\varphi(t,x)$ as $t \uparrow 0$. To this end, we write $u(t,x) = \varphi(t,x) + v(t,x)$. Then the equation that $v = v(t,x)$ satisfies is

$$\begin{cases} i\partial_t v = -\frac{1}{2}\partial_x^2 v - \frac{1}{2}\partial_x^2\varphi + (\lambda + i\kappa)(\mathcal{N}(\varphi + v) - \mathcal{N}(\varphi)), \\ v(0,x) = 0, \end{cases} \tag{9}$$

where $\mathcal{N}(u) = |u|^{p-1}u$. One may first suppose to apply the contraction mapping priciple to (9) via Duhamel's priciple. But this approach will not work so well, since the nonlinear estimate such as

$$|\mathcal{N}(\varphi + v) - \mathcal{N}(\varphi)| \leq C(|\varphi|^{p-1} + |v|^{p-1})|v|$$

contains the non-integrable singularity on $|\varphi|^{p-1} = O(|t|^{-1})$ around $t = 0$. Thus we need to apply another approach so called the energy method. To derive later a decay estimate of $\|u(t,\cdot)\|_{L^2(\mathbb{R})}$ as $t \to -\infty$, we must solve (9) in the weighted L^2 space. In this section, we have the next proposition.

Proposition 3.1. *Let $2 < p$, and let λ, κ satisfy* (2). *Then, for some $T_0 < 0$, there exists a unique solution $v = v(t,x)$ to* (9) *such that*

$$v \in C([T_0, 0]; H^1(\mathbb{R})) \cap C^1([T_0, 0); H^{-1}(\mathbb{R})), \tag{10}$$

$$xv \in C([T_0, 0]; L^2(\mathbb{R})). \tag{11}$$

Furthermore the solution satisfies

$$\|v(t,\cdot)\|_{L^2(\mathbb{R})} \le C|t|^{\alpha_0}, \quad \|\partial_x v(t,\cdot)\|_{L^2(\mathbb{R})} \le C|t|^{\alpha_1}, \tag{12}$$

where $\alpha_0 = 1 - 1/(p-1) - 2/(2N) > 0$ and $\alpha_1 = 1 - 1/(p-1) - 3/(2N) > 0$ with N defined in (A.4).

4. Proof of Theorem 1.1

We need to prolong the solution $u = \varphi + v$ backward in negative time. It is easy to guess that the size of the solution tends to 0 as $t \to -\infty$, since the nonlinear amplification (i.e., $\kappa > 0$) works as the dissipation in negative time direction. However this observation fails when $3 < p$ since the dispersion caused by $-(1/2)\partial_x^2$ turns down the nonlinearity. Hence the condition $p \le 3$ is required to ensure $\lim_{t\to-\infty} \|u(t)\|_{L^2(\mathbb{R})} = 0$.

Proposition 4.1. *Let $2 < p \le 3$ and λ, κ satisfy* (2). *Let $u(T_0,\cdot) \in H^1(\mathbb{R})$ and $xu(T_0,\cdot) \in L^2(\mathbb{R})$. Then the solution $u = u(t,x)$ to* (1) *exists globally in negative time. Furthermore we have*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \le C \begin{cases} (\log|t|)^{-1/3} & (p = 3), \\ |t|^{-(2/3)(1/(p-1)-1/2)} & (2 < p < 3) \end{cases} \tag{13}$$

for $t \in (-\infty, T_0]$.

Proposition 4.1 is related with the decay estimate of solutions. Such a problem has been of interest for dissipative nonlinear Schrödinger equations (DNLS). Shimomura [15] firstly derived an L^∞-decay of small-amplitude-solutions to DNLS with a cubic nonlinearity. It was extended to the sub-critical nonlinearity [12], to the large initial data [9, 11] and to the higher space dimension [2, 4, 5, 6]. The L^2-decay of solutions has been considered in [8, 13, 14]. We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1 By Proposition 3.1, there exists a solution to (1) in $[T_0, 0]$ such as $u(t,x) = \varphi(t,x) + v(t,x)$ where $\varphi(t,x)$ denotes a blowing-up profile determined in § 2 and $v(t,x)$ satisfies $v(0,x) = 0$. Since $u(T_0,\cdot) \in H^1(\mathbb{R})$ and $xu(T_0,\cdot) \in L^2(\mathbb{R})$, Proposition 4.1 is applied, and so we have a solution such that $\lim_{t\to-\infty} \|u(t)\|_{L^2(\mathbb{R})} = 0$. This means that, for any $\rho > 0$, there exists some $\tau < 0$ such that $\|u(\tau,\cdot)\|_{L^2(\mathbb{R})} < \rho$. Take $u(\tau,x) = u_0(x)$ as an initial datum of (1), and consider the positive time direction. Then, from the translation-invariance of (1) with respect to t and the uniqueness of the solution in $H^1(\mathbb{R})$, it follows that the solution u blows up at some $T^*(= |\tau|)$. □

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