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Article

A note on self-commutators of Volterra operator and its square

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Abstract. Let V be the classical Volterra operator on $L^2(0,1)$. The quadratic forms and their applications are used in many branches of mathematics. In this area of mathematics have studied many properties for concrete operator such as the numerical range. In recent year, the concept of Volterra operator has attached the serious attention of many researchers. In this paper, we compute the operator norm of self-commutators of V and its square. We also study the length of arc and the area of numerical range of V.

Keywords: Volterra operator, operator norm, self-commutator

1 Introduction

Let H be a complex Hilbert space equipped with the inner product $(\cdot\,;\cdot)$, which induces the norm $\|\cdot\|$. Denote by $B(H)$ the Banach algebra of bounded linear operators acting on H with the operator norm defined by

$$\|A\| = \sup_{\|x\|=1} \{\|A\| : x \in H\}, \quad A \in B(H).$$

For any compact $A \in B(H)$ the singular numbers $s_k(A)$ are the distances from A to the set of all operators of rank less than or equal to $k-1$; $k \geq 1$. Their squares are the eigenvalues of the compact self – adjoint nonnegative operator A^*A counted according to their multiplicities. In particular, $s_1(A) = \|A\|$.
We denote by V the classical Volterra operator

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$$(Vf)(x) = \int_0^x f(s)ds, \quad 0 < x < 1$$

on $L^2(0,1)$.

The later has been used by Halmos (see [3]) to calculate the

$$s_k(V) = \frac{2}{(2k-1)\pi}$$

for all $k \geq 1$, and $||V|| = \frac{2}{\pi}$.

The adjoint operator is

$$(V^* f)(x) = \int_x^1 f(s)ds.$$

The Volterra operator is compact, quasi-nilpotent, but not nilpotent. (see e.g. [1], [3])

We recall the well-known formula

$$(V^n f)(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt$$

and

$$(V^{*n} f)(x) = \int_x^1 \frac{(t-x)^{n-1}}{(n-1)!} f(t)dt$$

for $n \in \mathbb{N}$.

For a bounded linear operator A on a complex Hilbert space H, the numerical range $W(A)$ is the image of the unit sphere of H under the quadratic form $x \to (Ax, x)$ associated with the operator. More precisely,

$$W(A) = \{(Ax, x) : x \in H, \|x\| = 1\}.$$

It is well known that the numerical range of an operator is convex and the spectrum is contained in the closure of its numerical range. (see e.g. [1], [2], [3])

The difference $AA^* - A^*A = [A, A^*]$ is said to be the self-commutator of the operator A.

We shall need the following theorem.

Theorem 1. *(see [1, pp. 268]) If A is a bounded operator on H and $\varphi \in [-\pi, \pi]$, put $\lambda_\varphi = \max \sigma(B_\varphi)$, where $B_\varphi = \frac{1}{2}(e^{-i\varphi}A + e^{i\varphi}A^*) = B_\varphi^*$, then*

$$\overline{W(A)} = \bigcap_{\varphi \in [-\pi, \pi]} H_\varphi,$$

where the half-space H_φ is defined by

$$H_\varphi = \{z \in \mathbb{C} : Re(e^{-i\varphi} z) \leq \lambda_\varphi\}, \;\; z = x + iy.$$

Firstly, using the above theorem we show that the numerical range of V and its square on $L^2(0,1)$. (see [3], [5]) Researchers have studied extensively on numerical range for some powers V. Various results of operator norm, the numerical range for Volterra pencils have been presented in e.g. [4], [5], [6], [7] and [8].

The aim of this paper is to study the operator norm of the self-commutators of V and its square on $L^2(0,1)$. Also, we compute the length of arc and the area of a numerical range for V. Perhaps the above techniques could be applied for Volterra square operator.

2 The Results

Proposition 1. *([7]) According to Theorem 1 and under the assumption $\lambda_\varphi \in C^1[-\pi, \pi]$, we have*

$$\begin{cases} x = \lambda_\varphi \cos\varphi - \lambda'_\varphi \sin\varphi \\ y = \lambda_\varphi \sin\varphi + \lambda'_\varphi \cos\varphi \end{cases} \tag{1}$$

which is an envelope curve.

Proposition 2. *The length of arc of a $W(V)$ is $-\frac{1}{\pi} + Si(\pi) \approx 1,5336$, where $Si(x)$ is the integral sine function $Si(x) = \int\limits_0^x \frac{\sin t}{t} dt$.*

Proof. Note that $W(V)$ is bounded by the curve

$$\varphi \mapsto \frac{1-\cos\varphi}{\varphi^2} \pm i\frac{\varphi - \sin\varphi}{\varphi^2}, \quad \varphi \in [0, 2\pi].$$

We have

$$L = 2\int\limits_0^{2\pi} \sqrt{x'^2 + y'^2} d\varphi = 2\sqrt{2}\int\limits_0^{2\pi} \frac{\sqrt{4 + \varphi^2 - (4-\varphi^2)\cos\varphi - 4\varphi\sin\varphi}}{\varphi^3} d\varphi =$$

$$= 4\int_0^{2\pi} \left(2\sin\frac{\varphi}{2} - \varphi\cos\frac{\varphi}{2}\right) \frac{d}{d\varphi}\left(-\frac{1}{2}\varphi^{-2}\right) d\varphi.$$

By the integration by parts, we get

$$4\left\{\left(2\sin\left(\frac{\varphi}{2}\right) - \varphi\cos\left(\frac{\varphi}{2}\right)\right)\left(-\frac{1}{2}\varphi^{-2}\right)\Big|_0^{2\pi} - \int_0^{2\pi} \frac{\varphi}{2}\sin\left(\frac{\varphi}{2}\right)\left(-\frac{1}{2}\varphi^{-2}\right) d\varphi\right\} =$$

$$= -\frac{1}{\pi} + \int\limits_0^{\pi} \frac{\sin\varphi}{\varphi} d\varphi = -\frac{1}{\pi} + Si(\pi) \approx 1,5336.$$

Proposition 3. *The area of a $W(V)$ is*

$$\frac{1}{6}\left(\frac{1}{2\pi}+Si(2\pi)\right)\approx 0,2628.$$

Proof. We have

$$S=\int\limits_0^{2\pi}(xy^{'}-yx^{'})d\varphi=\int\limits_0^{2\pi}\frac{2(1-\cos\varphi)-\varphi\sin\varphi}{\varphi^4}d\varphi=$$

$$-\frac{1}{3}\int\limits_0^{2\pi}(2(1-\cos\varphi)-\varphi\sin\varphi)d\varphi^{-3}=$$

$$-\frac{1}{3}\left[\frac{2(1-\cos\varphi)-\varphi\sin\varphi}{\varphi^3}\Big|_0^{2\pi}-\int\limits_0^{2\pi}\frac{\sin\varphi-\varphi\cos\varphi}{\varphi^3}d\varphi\right]=$$

$$-\frac{1}{6}\int\limits_0^{2\pi}(\sin\varphi-\varphi\cos\varphi)d\varphi^{-2}=-\frac{1}{6}\left[\frac{\sin\varphi-\varphi\cos\varphi}{\varphi^2}\Big|_0^{2\pi}-\int\limits_0^{2\pi}\frac{\varphi\sin\varphi}{\varphi^2}d\varphi\right]=$$

$$=\frac{1}{6}\left(\int\limits_0^{2\pi}\frac{\sin\varphi}{\varphi}d\varphi+\frac{1}{2\pi}\right)=\frac{1}{6}\left(\frac{1}{2\pi}+Si(2\pi)\right)\approx 0,2628.$$

Proposition 4. *The operator norm of self-commutator for V is*

$$||[V,V^*]||=\frac{\sqrt{3}}{6}\approx 0,2886.$$

Proof. Put $A=VV^*-V^*V=[V,V^*]$. The spectral problem

$$Af=\lambda f$$

that is

$$\int_0^x\int_s^1 f(t)dtds-\int_x^1\int_0^s f(t)dtds=\lambda f(x).\ \ (\lambda\neq 0) \tag{2}$$

Let $D=\dfrac{d}{dx}$ denote the differential operator. Note that $DV=I$ and $DV^*=-I$. Applying D to the Eq.(2), we obtain the differential equation

$$\int_0^1 f(s)ds=\lambda f^{'}(x). \tag{3}$$

We insert $x=0$ and $x=1$ into Eq.(2) and simplifying, we get

$$\lambda^2=\frac{1}{12}. \tag{4}$$

Therefore, maximum of eigenvalue of the Eq.(4) is

$$||[V,V^*]||=\frac{\sqrt{3}}{6}\approx 0,2886.$$

Theorem 2. *The operator norm of self-commutator is for V^2 is*

$$||[V^2, V^{*2}]|| \approx 0,0799.$$

Proof. Put $A = V^2V^{*2} - V^{*2}V^2 = [V^2, V^{*2}]$. The spectral problem is

$$\int_0^x (x-s)ds \int_s^1 (t-s)f(t)dt - \int_x^1 (s-x)ds \int_0^s (s-t)f(t)dt = \lambda f(x). \tag{5}$$

We proceed from this integral equation to a differential equation by applying operator D^4 to Eq.(5). Thus

$$\int_0^x ds \int_s^1 (t-s)f(t)dt + \int_x^1 ds \int_0^s (s-t)f(t)dt = \lambda f^{'}(x) \tag{6}$$

$$\int_0^1 (t-x)f(t)dt = \lambda f^{''}(x) \tag{7}$$

$$\int_0^1 f(t)dt = -\lambda f^{'''}(x) \tag{8}$$

and

$$f^{(iv)}(x) = 0. \tag{9}$$

The general solution to Eq.(9) is

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \tag{10}$$

where a_0, a_1, a_2 and a_3 are arbitrary constants.

We insert $x = 0$ into Eq.(5), (6), (7) and (8), we get

$$\begin{cases} \frac{3}{4}a_0 + \frac{1}{5}a_1 + \frac{1}{12}a_2 + \frac{3}{70}a_3 = -6\lambda a_0 \\ \frac{1}{3}a_0 + \frac{1}{12}a_1 + \frac{1}{30}a_2 + \frac{1}{60}a_3 = 2\lambda a_1 \\ \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 = 4\lambda a_2 \\ a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 = -6\lambda a_3 \end{cases} \tag{11}$$

Therefore, its sufficient to see that maximum of eigenvalue for absolute value the following matrix

$$\begin{pmatrix} -\frac{1}{8} & -\frac{1}{30} & -\frac{1}{72} & -\frac{1}{140} \\ \frac{1}{6} & \frac{1}{24} & \frac{1}{60} & \frac{1}{120} \\ \frac{1}{8} & \frac{1}{12} & \frac{1}{16} & \frac{1}{20} \\ -\frac{1}{6} & -\frac{1}{12} & -\frac{1}{18} & -\frac{1}{24} \end{pmatrix} \tag{12}$$

The characteristic equation of (12) is

$$\lambda^4 + \frac{1}{16}\lambda^3 - \frac{11}{8064}\lambda^2 + \frac{13}{4838400}\lambda + \frac{1}{1741824000} = 0$$

Thus, the maximum of eigenvalue for absolute value of the Eq.(12) is

$$|||[V^2, V^{*2}]||| = \max|\lambda| \approx 0,0799.$$

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