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On the properties of Bäcklund transformations on Painlevé 6th equations

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Abstract. In his series of papers Okamoto give detailed explanation of Painlevé equations of types 2-6. On the other hand Kajiwara et.al described Bäcklund transformations for these types of Painlevé equations. Later, Bobenko and Eitner found that Painlevé equations of types 6 and 5 give rise Bonnet surfaces and vise-versa. In this research we study properties of Bäcklund transformations of 6th Painlevé equations P_6 and determine explicit formula of such transformations which give a new Bonnet surfaces.

Keywords: Painlevé equations, Bäcklund transformations, Affine Weyl group, Bonnet surfaces

1 Introduction

Consider an algebraic differential equation of order n

$$F\left(t, y, \frac{dy}{dt}, \dots \frac{d^n y}{dt^n}\right) \tag{1}$$

where $F(t, y_0, \dots, y_n)$ is a polynomial in $y_0, \dots, y_n$ and holomorphic in $t \in U$ for some domain $U \subset \mathbf{C}$.

Let $y(t) = y(t, c)$ be a solution of (1) with an initial condition

$$\frac{d^i y}{dt^i}(t_0) = c_i, i = 0, \dots, n-1 \tag{2}$$

where $t_0 \in U$ and $c = (c_0, \dots c_{n-1}) \in \mathbf{C}^n$ are initial data.

First we define a Painlevé property for differential equations.

Definition 1. *A differential equation of form* (2) *have the Painlevé property if it's moveable singularities are only poles.*

This means that poles are only singularities of the solutions of (2) which change their position when initial data $c \in \mathbf{C}^n$ vary.

The case when the equation (1) is non-linear is most challenging one and the reason is related with singular points.

The French mathematician P.Painlevé and later his students studied the problem of classifying such types of nonlinear differential equations of second order with Painlevé

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properties. They studied six types of Painlevé equations and their solutions are called Painlevé transcendents. Here we give detailed list of Painlevé equations.

$$
\begin{aligned}
P_1 :& y'' = 6y^2 + t \\
P_2 :& y'' = 2y^3 + ty + \alpha \\
P_3 :& y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_4 :& y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
P_5 :& y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2} \\
&\qquad \cdot \left(\alpha y + \frac{\beta}{t}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1} \\
P_6 :& y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 \\
&- \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
&+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2}\right. \\
&\left. + \delta\frac{t(t-1)}{(y-t)^2}\right)
\end{aligned}
$$

For all of these cases α, β, γ and δ are complex numbers.

In his series of papers [5]- [8], Okamoto gave detailed description of Painlevé equations.

Definition 2. *Let F be a smooth surface in $\mathbf{R}^3$ with non-constant mean curvature. The surface F is called Bonnet surface, if there exists a one-parameter family of surfaces*

$$F_\tau, \tau \ni (-\varepsilon, \varepsilon) > 0, F_0 = F$$

of non trivial isometric deformations preserving the mean curvature function. The family $(F_\tau)_{\tau \ni (-\varepsilon,\varepsilon)}$ is called Bonnet family.

The solutions of Painlevé equations of types P_{VI}, P_V and P_{III} gives Bonnet surfaces of so called types A, B and C, respectively ([1]).

For this reason, the study of properties of solutions of Painlevé equations specially the cases of P_{VI}, P_V and P_{III} is very important in the theory of Bonnet surfaces.

Roughly speaking, a Bäcklund transformation of Painlevé equations are those which transforms solutions of a certain Painlevé equation with given parameters to a new solution with different parameters.

However, the Bäcklund transformations for Painlevé equations are defined it is complicated for further use and calculations. Moreover construction of Bonnet surfaces from solutions of Painlevé equations of P_{VI}, P_V and P_{III} is still hard problem. The aim of this paper is to study properties of Bäcklund transformations which will help to simplify further calculations for the case of P_{VI}. Moreover we define explicit Bäcklund transformations which give Bonnet surfaces.

2 Sixth Painlevé equations and their Bäcklund transformations

For the sixth Painlevé equation P_6 we take Hamiltonian

$$\begin{aligned} H = & \frac{1}{t(t-1)}\big[q(q-1)(q-t)p^2 - [\kappa_0(q-1)(q-t) \\ & + \kappa_1 q(q-t) + (\theta-1)q(q-1)]p \\ & + \kappa(q-t)\big], \end{aligned}$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}\kappa_\infty, \beta = -\frac{1}{2}\kappa_0^2, \gamma = \frac{1}{2}\kappa_1^2, \delta = \frac{1}{2}(1-\theta^2), \\ \kappa &= \frac{1}{4}(\kappa_0^2+\kappa_1^2+\theta-1)^2 - \frac{1}{2}\kappa_\infty, \end{aligned}$$

then we have a Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

As we stated before, the Bäcklund transformation of Painlevé equations P_{VI} transforms solutions of a equation with given parameters to a solution of a new P_{VI} equation with different parameters.

In [3] and [2], it given Bäcklund transformations of Painlevé equation P_{VI}. Usually, the Painlevé equations (except for PI) admit two classes of classical solutions. One and rather simple one are algebraic or rational solutions. The another one is a class of classical transcendental solutions expressed in terms of special functions of hypergeometric type. For the case of P_{VI} all the classes are well studied in [5], [3] and [2].

For example, Okamoto gave the examples of rational solutions of P_{VI} are given in as follows([5])

$$(q_m, p_m) = \left(\frac{m+1}{t+m}, \frac{t+m}{t+m+1}\right).$$

We set

$$\alpha_0 = \theta, \alpha_1 = k_\infty, \alpha_3 = k_1, \alpha_4 = k_0$$

and

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1.$$

For simplicity of notation we define maps s_i, $i = 0, ..., 4$

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, j = 0, ..., 4, \tag{3}$$

	α_0	α_1	α_2	α_3	α_4	f_4	f_2
s_0	$-\alpha_0$	α_1	$\alpha_2+\alpha_0$	α_3	α_4	f_4	$f_2-\dfrac{\alpha_0}{f_0}$
s_1	α_0	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	α_4	f_4	f_2
s_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$\alpha_4+\alpha_2$	$f_4+\dfrac{\alpha_2}{f_2}$	f_2
s_3	α_0	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	α_4	f_4	$f_2-\dfrac{\alpha_3}{f_3}$
s_4	α_0	α_1	$\alpha_2+\alpha_4$	α_3	$-\alpha_4$	f_4	$f_2-\dfrac{\alpha_4}{f_4}$
s_5	α_1	α_0	α_2	α_4	α_3	$t\dfrac{f_3}{f_0}$	$-\dfrac{f_0(f_2f_0+\alpha_2)}{t(t-1)}$
s_6	α_3	α_4	α_2	α_0	α_1	$\dfrac{t}{f_4}$	$-\dfrac{f_4(f_4f_2+\alpha_2)}{t}$
s_7	α_4	α_3	α_2	α_1	α_0	$\dfrac{f_0}{f_3}$	$\dfrac{f_3(f_3f_2+\alpha_2)}{t-1}$

where the matrix $A=(a_{ij})\,,\, i,j=0,...,4$ is defined as follows.

$$A=\begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Please refer the following table for transformations for coefficients and variables of P_{VI} ([2] and [3]).

where

$$f_0=q-t, f_3=q-1, f_4=q_4, \text{ and } f_2=p.$$

3 Properties of the Bäcklund transformations

Let $s_i, i=0,...,4$ be fundamental Bäcklund transformations of P_{VI} as defined in (3). A discrete group $W=<s_0,s_1,\dots,s_4>$ is an affine Weyl group of a root system D_4 and we can define an extended affine Weyl group by $\widetilde{W}=<s_0,\dots,s_7>$. The extended affine Weyl group acts on Painlevé equations P_6 and these are Bäcklund transformations for P_6. In [3], Masuda determined folowing Bäcklund transformations and studied their certain properties of actions for tau-functions:

$$\begin{aligned} &T_{13}=s_1s_2s_0s_4s_2s_1s_7, T_{40}=s_4s_2s_1s_3s_2s_4s_7,\\ &T_{34}=s_3s_2s_0s_1s_2s_3s_5, T_{14}=s_1s_4s_2s_0s_3s_2s_6, \end{aligned} \tag{4}$$

These transformations act on $\alpha_i, i=0,\dots,4$ as:

$$\begin{aligned} T_{13}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) =&(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)+(0,1,0,-1,0)\\ T_{40}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) =&(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)+(-1,0,0,0,1)\\ T_{34}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) =&(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)+(0,0,0,1,-1)\\ T_{14}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) =&(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)+(0,1,-1,0,1) \end{aligned}$$

We refer [4] for the case of P_2 and P_4.

In this section we derive some properties of these transformations.

First we check conditions for

$$(s_j s_i)^\ell = 1,$$

that is

$$s_j(s_i(...(s_j(s_i(\alpha_k)))...)) = \alpha_k.$$

We simplify $s_i s_j(\alpha_k)$ for $0 \le k, j, i \le 4$.

By virtue of the definition (3),

$$\begin{aligned} s_j(s_i(\alpha_k)) =& s_j(\alpha_k - a_{ik}\alpha_i) \\ =& \alpha_k - a_{jk}\alpha_j - a_{ik}(\alpha_i - a_{ji}\alpha_j) \end{aligned}$$

and

$$\begin{aligned} (s_j s_i)^2(\alpha_k) =& s_j(s_i(\alpha_k - a_{jk}\alpha_j - a_{ik}(\alpha_i - a_{ji}\alpha_j))) \\ =& s_j(\alpha_k - a_{ik}\alpha_i - a_{jk}(\alpha_j - a_{ij}\alpha_i) - a_{ik}(-\alpha_i) \\ &+ a_{ik}a_{ji}(\alpha_j - a_{ij}\alpha_i)) \\ =& \alpha_k - a_{jk}\alpha_j - a_{ik}(\alpha_i - a_{ji}\alpha_j) - a_{jk}(-\alpha_j) \\ &+ a_{jk}a_{ij}(\alpha_i - a_{ij}\alpha_j) + a_{ik}(\alpha_i - a_{ij}\alpha_j) \\ &- a_{ik}a_{ji}\alpha_j - a_{ik}a_{ji}a_{ij}(\alpha_i - a_{ij}\alpha_j) \\ =& \alpha_k + (-a_{jk}a_{ij}^2 - a_{ik}a_{ji} + a_{ik}a_{ij}^3)\alpha_j \\ &+ (a_{jk}a_{ij} - a_{ik}a_{ij}^2)\alpha_i. \end{aligned}$$

Hence

$$(s_j s_i)^2(\alpha_k) = \alpha_k$$

only when

$$\begin{aligned} -a_{jk}a_{ij}^2 - a_{ik}a_{ji} + a_{ik}a_{ij}^3 = 0 \text{ and} \\ a_{jk}a_{ij} - a_{ik}a_{ij}^2 = 0. \end{aligned} \tag{5}$$

We can check that the conditions (5) hold for all $0 \le i, j \le 4$ with

$$(i, j) \ne (0, 2), (1, 2), (2, 3), \text{ and } (2, 4).$$

We also can check that

$$(s_0 s_2)^3 = 1, (s_1 s_2)^3 = 1, (s_2 s_3)^3 = 1, \text{ and } (s_2 s_4)^3 = 1. \tag{6}$$

Now we formulate our main result as follows.

Proposition 1. *For fundamental Bäcklund transformations for Painlevé equations P_6 the following relations hold.*

1. $s_i^2 = 1$ *for all* $i = 0, \ldots, 7$.
2. $(s_i s_j)^2 = 1$ *for all* $0 \leq i, j \leq 4$, *except* $(i, j) \neq (0, 2), (1, 2), (2, 3),$ *and* $(2, 4)$.
3. $(s_0 s_2)^3 = (s_1 s_2)^3 = (s_2 s_4)^3 = 1$.
4. *For* $j = 5, 6, 7$, $(s_i s_j)^4 = 1$ *for* $i \neq 2$ *and* $(s_2 sj)^2 = 1$.

We note here that since $s_i^2 = 1$, $i = 0, \ldots 7$, we have $(s_i s_j)^k = (s_j s_i)^k$, for all $i, j = 0, \ldots 7$.

In [1], it considered following Painlevé equation:

$$\begin{aligned} y'' =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 \\ & - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y' \\ & + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\frac{\theta^2}{2}\frac{t-1}{(y-1)^2}\right. \\ & \left. - \frac{\theta(\theta+2)}{2}\frac{t(t-1)}{(y-t)^2}\right) \end{aligned} \tag{7}$$

and showed that the solutions of this equations give Bonnet surface of types A.

This is Painlevé equation of type P_{VI} where

$$\alpha = 0, \beta = 0, \gamma = \frac{\theta^2}{2} \text{ and } \delta = -\frac{\theta(\theta+2)}{2}. \tag{8}$$

Similarly, they also showed that solutions of Painlevé equations of types P_V and P_{III} give Bonnet surfaces of types B and C.

Now let us take composites of Bäcklund transformations defined in (9) for equations defined in (7).

First, note that using Proposition 1 we can define

$$\begin{aligned} T_{13}^{-1} &= s_7 s_1 s_2 s_4 s_0 s_2 s_1, \; T_{40}^{-1} = s_7 s_4 s_2 s_3 s_1 s_2 s_4, \\ T_{34}^{-1} &= s_5 s_3 s_2 s_1 s_0 s_2 s_3, \;\; T_{14}^{-1} = s_6 s_2 s_3 s_0 s_2 s_4 s_1 \end{aligned} \tag{9}$$

and consider the composite

$$T = T_{14}^n T_{34}^m T_{40}^\ell T_{13}^k. \tag{10}$$

Then, the general formula for this transformation can be given as

$$\begin{aligned} T(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) =& (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ & + (-\ell, k+n, -n, m-k, \ell-m+n). \end{aligned}$$

The transformation T will act on the equation (7) as follows

$$\begin{aligned} T(-2-r-r, 1+r, 1, 0) =& (-2-r, -r, 1+r, 1, 0) \\ & + (-\ell, k+n, -n, m-k, \ell-m+n). \end{aligned} \tag{11}$$

If we assume that the solutions of equations derived from the transformation (11) also give a Bonnet surface, the conditions (8) give us

$$\begin{aligned} -r+k+n&=0 \\ \ell-m+n&=0 \\ \frac{1}{2}(1+m-k)^2&=\frac{1}{2}\overline{\theta}^2 \\ \frac{1}{2}(1-(-2-r-\ell)^2)&=\frac{1}{2}\overline{\theta}(\overline{\theta}+2) \end{aligned} \tag{12}$$

From first two identities, we deduce

$$\ell+r=m+k$$

and using this into last two identities we have

$$\begin{aligned} \overline{\theta}&=\pm(1+m-k) \\ (1+m-k)^2+2\overline{\theta}&=-(1+m+k)(3+m+k). \end{aligned}$$

We look the cases $\overline{\theta}=1+m-k$ and $\overline{\theta}=-(1+m-k)$ separately.

3.1 The case $\overline{\theta}=1+m-k$

It will lead to the condition $k^2+m^2+4m+3=0$ which is true only when $-3\leq m\leq 1$. This equation has following solutions (m,k) in $\boldsymbol{Z}$:

$$(-1,0),(-2,\pm 1), \text{ and } (-3,0).$$

Using these we can find quintuples (r,n,m,ℓ,k) which satisfy (12) expressed in n as follows:

$$\begin{aligned} &(n,n,-1,-n-1,0) \\ &(n,n,-3,-n-3,0) \\ &(n\pm 1,n,-2,-n-2,\pm 1) \end{aligned}$$

3.2 The case $\overline{\theta}=-(1+m-k)$

It will lead to the condition $k^2+k+(1+m)^2=0$ which is true only when $-1\leq k\leq 0$. This equation has following solutions (m,k) in $\boldsymbol{Z}$:

$$(-1,0) \text{ and } (-1,-1).$$

Using these we can find quintuples (r,n,m,ℓ,k) which satisfy (12) expressed in n as follows:

$$\begin{aligned} &(n,n,-1,-n-1,0) \\ &(n-1,n,-1,-n-1,-1) \end{aligned}$$

We sum up all the cases with the following Proposition.

Proposition 2. *Painlevé equations* (7) *transformed by following Bäcklund transformations and solutions of a new equations give Bonnet surfaces:*

$$\begin{aligned}
T &= T_{14}^{n}T_{34}^{-1}T_{40}^{-n-1}T_{13}^{0}\\
T &= T_{14}^{n}T_{34}^{-3}T_{40}^{-n-3}T_{13}^{0}\\
T &= T_{14}^{n}T_{34}^{-2}T_{40}^{-n-2}T_{13}^{1}\\
T &= T_{14}^{n}T_{34}^{-2}T_{40}^{-n-2}T_{13}^{-1}\\
T &= T_{14}^{n}T_{34}^{-2}T_{40}^{-n-2}T_{13}^{1}\\
T &= T_{14}^{n}T_{34}^{-2}T_{40}^{-n-2}T_{13}^{-1}
\end{aligned} \tag{13}$$

4 Conclusion

Finding an explicit form of images of solutions of the Painlevé equations which give a rise to Bonnet surfaces by Bäcklund transformations given by the formula in (13) is a difficult problem since it expressed in tau functions. Simple examples of such solutions including rational ones and their images by the Bäcklund transformations will be very useful to understand the whole picture. This will be a topic of our future studies.

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References

1. A. Bobenko and U.Eitner, Painlevé equations in the differential geometry, Lecture notes in mathematics, 1753, (2000). https://doi.org/10.1007/b76883
2. K. Kajiwara, T.Masuda, M.Noumi, Y.Ohta and Y.Yamada, Determinant formulas for the Toda and discrete Toda equations, Funkcial. Ekvac. 44 (2001), pp. 291–307.
3. T.Masuda, Classical transcendental solutions of the Painlevé equations and their degeneration, Tohoku Math. J. 56 (2004), pp. 467–490. https://doi.org/10.2748/tmj/1113246745
4. M.Noumi, Painlevé equations through symmetry, Translations of mathematical monographs, 2004. https://doi.org/10.1090/mmono/223
5. K.Okamoto, Studies on the Painlevé equations I, sixth Painlevé equation PVI, Ann. Mat. Pura Appl. 4- 146 (1987), pp. 337–381. https://doi.org/10.1007/BF01762370
6. K.Okamoto, Studies on the Painlevé equations II, fifth Painlevé equation PV, Japan J. Math. 13 (1987), pp. 47–76. https://doi.org/10.4099/math1924.13.47
7. K.Okamoto, Studies on the Painlevé equations III, second and fourth Painlevé equations, PII and PIV, Math. Ann. 275 (1986), pp. 222–254. https://doi.org/10.1007/BF01458459
8. K.Okamoto, Studies on the Painlevé equations IV, third Painlevé equation PIII, Funkcial. Ekvac. 30 (1987), pp. 305–332.