Journal of Applied Science and Engineering A,
Volume 4, Issue 1, 2023, pp.01-07.
https://doi.org/10.5564/jasea.v4i1.2684

Article

Multiplicative optimal control problem

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Abstract. In this paper, we consider a multiplicative optimal control problem subject to a system of linear differential equation.It has been shown that product of two concave functions defined positively over a feasible set is quasiconcave. It allows us to consider the original problem from a view point of quasiconvex maximization theory and algorithm. Global optimality conditions use level set of the objective function and convex programming as subproblem. The objective function is product of two concave functions. We consider minimization of the objective functional. The problem is nonconvex optimal control and application of Pontriyagin's principle does not always guarantee finding a global optimal control. Based on global optimality conditions, we develop an algorithm for solving the minimization problem globally.

Keywords: Pontriyagin's principle, quasiconcave, quasiconvex

1 Introduction

We consider the following multiplicative optimal control minimization problem:

$$\min_{u\in V} f(x(T))\cdot g(x(T)) \tag{1.1}$$

$$\begin{cases} \dot{x}(t) = A(t)x + B(t)u(t) + C(t) \\ x(t_0) = x^0 \end{cases} \tag{1.2}$$

$$u \in V = \{u \in L_2^r([t_0,T]) | u(t) \in U, \ \ t \in [t_0,T]\} \tag{1.3}$$

where t_0 and T are given with $-\infty < t_0 < T < +\infty$, $x = [x_1, ..., x_n]^T \in R^n$, $u(t) \in [u_1(t), u_2(t), ..., u_r(t)]^T \in R^r$ are respectively, the state and control, and elements of the matrix valued functions $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times r}$ and $C(t) \in R^{n\times 1}$ are piecewise continuous on $[t_0, T]$. Let $U \subset R^r$ be a compact and convex subset. The above problem has many applications in engineering and economics. For instance, a problems of maximizing advertising efficiency [13] and an efficiency of average productivity are formulated as a multiple programming. There are numerous methods in the literature for solving problem (1.1)–(1.3) in a finite dimensional space. Problem (1.1)–(1.3) has been considered in a finite dimensional case in [2, 3, 7, 11, 15, 18, 22] for the case when f is concave and g is convex. We formulate problem(1.1) as a terminal multiplicative nonconvex optimal control and then we reduce it to a quasiconvex maximization so that we could apply a result in [5].

We call problem (1.1)–(1.3) as the multiplicative optimal control minimization problem. It is well known that [17, 19, 8] the solution of system (1.2) can be written as:

Received: 20 December 2022; Revised: 17 March 2023; Accepted: 24 April 2023

$$x\,(u,t\,) = F(t,t_0)x^0 + \int_{t_0}^{t} F(t,\tau)\,[B(\tau)u(\tau) + C(\tau)]\,d\tau \tag{1.4}$$

where, $F(t,\tau) \in R^{n\times n}$ is the fundamental matrix solution of the matrix equation

$$\begin{cases} \frac{\partial F(t,\tau)}{\partial t} = A(t)F(t,\tau), \quad t \geq \tau \in [t_0,T] \\ F(\tau,\tau) = I \end{cases} \tag{1.5}$$

Here, I denotes the identity matrix. Note that $x\,(u,t\,)$ is an absolutely continuous vector-valued function of the time t. Define the reachable set of system (1.2) with respect to $u \in U$.

$$D = D(T) = \{y \in R^n | y = x\,(u,t\,),\ u \in U\}\,. \tag{1.6}$$

It is known that $D \subset R^n$ is a convex set [19]. Then multiple optimal minimization control problem can written as

$$\min_{x\in D} \varphi(x(T)) = f(x(T)) \cdot g(x(T)). \tag{1.7}$$

Finally, assume that $f, g : D \to R$ are concave on D. Also, $f(\cdot\,)$ and $g(\cdot\,)$ are supposed to be differentiable and positive defined on D.
The rest of the paper is organized as follows. Multiple optimal control minimization problem with the linear controlled system of differential equations has been considered in Section 2. In Section 3, an algorithm based on approximation of reachable set is given.

2 Multiplicative optimal control minimization problem

Definition 1. *[4] A function $\varphi : \mathbb{D} \to \mathbb{R}$ is said to be quasiconcave on a convex set $\mathbb{D} \subset R^n$*

$$\varphi\,(\alpha x + (1-\alpha)y) \geq \min\{\varphi(x), \varphi(y)\}$$

is satisfied for all $x, y \in D$ and $\alpha \in [0,1]$. If φ is quasiconcave then $-\varphi$ is called quasiconvex.

Theorem 1. *[2] A function $f : \mathbb{D} \to \mathbb{R}$ is quasiconvace on $\mathbb{D}$ if and only if the set*

$$L_c(f) = \{x \in \mathbb{D} | \varphi(x) \geq c\}$$

is convex for all $c \in \mathbb{R}$.

Consider a problem of minimizing the product of two concave funtions

$$\min_{x\in\mathbb{D}} \varphi = f \cdot g$$

where $f, g : \mathbb{D} \to \mathbb{R}$ are positive defined concave functions on a convex set $\mathbb{D} \subset \mathbb{R}^n$.

Lemma 1. *[2] The function φ is quasiconcave on $\mathbb{D} \subset \mathbb{R}^n$.*

Proof. Define the set $L_c(\varphi)$:

$$L_c(\varphi) = \{x \in | \varphi(x) \geq c\}$$

for all positive $c \in R^+$.
We show that $L_c(\varphi)$ is convex. Take points $x, y \in L_c(\varphi)$ and $\alpha \in [0,1]$.
Then $\varphi(\alpha x + (1-\alpha)y) \geq \alpha\varphi(x) + (1-\alpha)\varphi(y)$ and means that $\alpha x + (1-\alpha)y \in L_c(\varphi), \quad c > 0$.
Now we are ready to formulate global optimality conditions for problem (1.7)
On the other hand, problem (1.7) can be treated equivalently, as a quasiconvex maximization problem

$$\min_{x \in \mathbb{D}}(\varphi) = -\max(-\varphi) = -max\ \overline{\varphi}(x) \tag{2.1}$$

where, $\overline{\varphi}(x) = -\varphi(x)$.
From Lemma 1, it is clear that the function $\varphi\left(x(\cdot)\right)$ is quasiconvex on D. Thus, problem (1.7) is a quasiconcave minimization problem while problem (2.1) is an equivalent quasiconvex maximization problem. Now, we shall apply the global optimality conditions [5] to Problem (2.1).

Theorem 2. [5] *Let*

$$E_{\overline{\varphi}(z)}(\overline{\varphi}) = \{y \in R^n | \overline{\varphi}(y) = \overline{\varphi}(z)\} \tag{2.2}$$

Conditions

$$\langle \overline{\varphi}'(y), x - y\rangle \ \leq 0 \tag{2.3}$$

holds for all $y \in E_{\overline{\varphi}(z)}(\overline{\varphi})$ and $x \in D$, where $\overline{\varphi}'$ denotes the gradient. In addition, if $\overline{\varphi}'(y) \neq 0$ hold for all $y \in E_{\overline{\varphi}(z)}(\overline{\varphi})$, then, condition (2.3) is a sufficient condition for $z \in D$ to be a global solution to problem (2.1).

Lemma 2. *Suppose that for any feasible points $x, y \in D$ such that the inequality*

$$\langle \overline{\varphi}'(y), x - y\rangle \ > 0$$

holds. Then, $\overline{\varphi}(x) \geq \overline{\varphi}(y)$.

Proof. On the contrary, assume that $\overline{\varphi}(x) < \overline{\varphi}(y)$. Since $\overline{\varphi}$ is quasiconvex, we have

$$\overline{\varphi}\left(\alpha x + (1-\alpha)y\right) \leq \max\left\{\overline{\varphi}(x), \overline{\varphi}(y)\right\} = \overline{\varphi}(y)$$

By Taylor's formula, there is a neighborhood of the point y on which

$$\overline{\varphi}\left(y + \alpha(x-y)\right) - \overline{\varphi}(y) = \alpha\left(\langle\overline{\varphi}'(y), x - y\rangle + \frac{o(\alpha\,\|x-y\|)}{\alpha}\right) \leq 0,$$

for sufficiently small $\alpha > 0$, where

$$\lim_{\alpha\to 0}\frac{o(\alpha\,\|x-y\|)}{\alpha} = 0.$$

Therefore, $\langle\overline{\varphi}'(y), x - y\rangle \ \leq 0$ which contradicts $\langle\overline{\varphi}'(y), x - y\rangle \ > 0$. This completes the proof.

Let u^* be an admissible control which is a global optimal control to problem (1.7) and let x^* be the corresponding solution of system (1.2). Introduce an auxiliary function $\Pi(y)$ defined by

$$\Pi(y) = \max_{x \in D}\langle\overline{\varphi}'(y), x - y\rangle, \quad y \in R^n \tag{2.4}$$

Then, based on Theorem 2, we can derive the global optimality conditions for Problem (1.7) in the following theorem.

Theorem 3. *A control $u^* \in V$ is a global optimal control to problem (1.7) if and only if*

$$\max\left\{\Pi(y)|y \in E_{\overline{\varphi}(x^*)}(\varphi)\right\} \le 0 \tag{2.5}$$

where $x^* = x^*(u^*, T) \in D(T)$.

Proof. The validity of Theorem 3 is equivalent to that of the optilimality condition (2.3).

From Theorem 3, we can conclude that if there exist a process $(\tilde{x}, \tilde{u})$ and $\tilde{y} \in E_{\varphi(\bar{x})}(\varphi)$ such that

$$\langle \varphi'(\tilde{y}), \tilde{x} - \tilde{y} \rangle > 0 \tag{2.6}$$

then the control $\bar{u}$ is not a global optimal control to problem (1.7), where $\tilde{x} = x\left(\tilde{u}, T\right)$, $\tilde{y} = y\left(\bar{u}, T\right)$ and $\tilde{u}, \bar{u} \in V$. Before we formulate an algorithm for solving problem (1.7), we need to compute $\Pi(y)$ for any $y \in R^n$. First, we consider the linear optimal control problem

$$\max_{x \in D} \langle \overline{\varphi}'(y), x \rangle. \tag{2.7}$$

Consider the following system of differential equations for a given $y \in R^n$.

$$\begin{cases} \dot{\psi} = -A^T \psi \\ \psi(T) = -\overline{\varphi}'(y) \end{cases} \tag{2.8}$$

This system, which is known as the adjoint system, has a unique piecewise differentiable solution $\psi(t) = y\left(y, t\right)$ defined on $[t_0, T]$, where $\psi(t, y) = \left[\psi_1(t), \cdots, \psi_n(t)\right]^T$. $\psi(t)$ is referred to as the adjoint variable. Problem (1.7) can be solved by using the results presented in the following theorem.

Theorem 4. [3] *Let $\psi(t) = \psi\left(y, t\right)$, $t \in [t_0, T]$ be a solution of the adjoint system (2.8) for $y \in R^n$. An admissible control $z(t) = z\left(y, t\right)$ is an optimal control to Problem (1.7), then it is necessary and sufficient that*

$$\langle \psi\left(y, t\right), B(t) z\left(y, t\right) \rangle = \min_{u \in V} \langle \psi\left(y, t\right), B(t) u(t) \rangle \tag{2.9}$$

for almost every $t \in [t_0, T]$.

On the basis of Theorem 4, the value $\Pi(y)$ can be computed by using the following algorithm.

Algorithm OPTLIN

1. Solve the adjoint system (2.8) for a given $y \in R^n$. Let $\psi(t) = \psi\left(y, t\right)$ be the solution.
2. Find the optimal control $z(t) = z\left(y, t\right)$ as a solution of the problem

$$\min_{u \in U} \langle \psi(t), B(t) u \rangle$$

at each moment of $t \in [t_0, T]$.

3. Find a solution $x(t) = x\left(z, t\right)$ of system (1.2) for $u(t) = z\left(y, t\right)$.
4. Find $x(T) = x\left(z, T\right)$ by (4) with $t = T$.
5. Compute $\Pi(y)$ by the formula $\Pi(y) = \langle \overline{\varphi}'(y), x(T) - y \rangle$.

3 Approximation set

We use the following definition introduced in [6].

Definition 2. *For a given integer m, let A_z^m be the set defined by*

$$A_z^m = \{y^1, y^2, \cdots, y^m | y^i \in E_{\overline{\varphi}(z)}(\overline{\varphi}) \cap D, i = 1, 2, \ldots, m\}$$

Then, it is called an approximation set, where $z = x(u, T)$, $u \in V$.

Lemma 3. *Suppose that there exist a feasible point $z \in D$ and a point $y^i \in A_z^m$ such that*

$$\langle \overline{\varphi}'(y^j), u^j - y^j \rangle \ > 0$$

then $\overline{\varphi}(u^j) > \overline{\varphi}(z)$, where $\langle \overline{\varphi}'(y^j), u^j \rangle \ = \max_{x \in D} \langle \overline{\varphi}'(y^j), x \rangle$.

Proof. The proof follows from Lemma 2.

Based on the properties of quasiconvexity of $\overline{\varphi}(\cdot)$ and global optimality conditions, we propose an algorithm for solving the problem (1.7). The algorithm which differs from Algorithm 2 [6] in finding a local optimal control may now be written as follows.

Algorithm OPTGL

Step 1. Let $k := 0$ and let $\bar{u}^k \in V$ be an arbitrary given control. Starting with the control $\bar{u}^k$, we find a local optimal control u^k by using the optimal control software OPTCON [9, 10].

Step 2. Find $x^k = x\left(u^k, T\right)$ by solving system (1.2) for $u = u^k$.

Step 3. Construct the approximation set $A_{x^k}^m$ as follows:

$$A_{x^k}^m = \left\{y^1, y^2, \cdots, y^m | y^i \in E_{\overline{\varphi}(x^k)}(\overline{\varphi}) \cap D(T), \ i = 1, 2, \ldots, m\right\}.$$

Step 4. Solve the linear optimal control problems

$$\max_{x \in D(T)} \langle \overline{\varphi}'(y^i), x \rangle, \quad i = 1, 2, \ldots, m.$$

Step 5. Compute $\Pi(y^i)$, $i = 1, 2, \ldots, m$, by Algorithm 1.

Step 6. Compute η_k:

$$\eta_k = \Pi(y^j) = \max_{1 \le i \le m} \Pi(y^i),$$

let $z^j = z^j(y^j, t)$ be the solution of the problem:

$$\langle \psi^j(t), B(t) z^j \rangle = \min_{u \in U} \langle \psi^j(t), B(t) u \rangle, t \in [t_0, T],$$

where

$$\begin{cases} \dot{\psi}^j(t) = -A^T(t) \psi^j(t) \\ \psi^j(T) = -\overline{\varphi}'(y^j) \end{cases}$$

Step 7. If $\eta_k \le 0$ then terminate. u^k is a global approximate solution; otherwise, go to next step.

Step 8. Set $\bar{u}^{k+1} := z^j(y^j, t)$ and $k := k + 1$. Then, go to step 2.

Lemma 4. *Suppose that there is a point* $y^j \in A^m_{x^k}$ *for* $u^k \in D(T)$ *such that* $\langle \overline{\varphi}'(y^j), x\left(z^j, T\right) - y^j \rangle > 0$, *where* z^j *satisfies* $\langle \overline{\varphi}'(y^j), x(z^j, T) \rangle = \max_{x(T) \in D(T)} \langle \overline{\varphi}'(y^j), x \rangle$. *Then, it holds that*

$$\overline{\varphi}\left(x(z^j, T)\right) > \overline{\varphi}\left(x^k(z^k, T)\right)$$

Proof. From Lemma 2, we have

$$\langle \overline{\varphi}'(y^j), x(z^j, T) - y^j \rangle > 0$$

thus

$$\overline{\varphi}\left(x(z^j, T)\right) \geq \overline{\varphi}\left(y^j\right) = \overline{\varphi}\left(x^k(u^k), T\right)$$

This completes the proof.

Theorem 5. *If* $\eta_k > 0$ *for all* $k = 1, 2, \ldots, s$, *the sequence* $\left\{J(u^k)\right\}$ *constructed by Algorithm OPTGL is a monotonic increasing sequence, i.e.,*

$$J(u^{k+1}) > J(u^k), \; k = 1, 2, \ldots, s$$

where $J(u^k) = \overline{\varphi}\left(x(u^k, T)\right)$.

4 Conclusions

Multiple optimal control minimization problem has been considered. The problem is nonconvex and reduces to a quasiconvex maximization problem in a finite dimensional space via the reachable set of the system. For solving the maximization problem we used the global optimality conditions [5]. We propose the Algorithm OPTGL based on these conditions. Subproblems of Algorithm OPTGL are linear optimal control problems which make the algorithm easily implementable. Numerical implementation will be discusseal in a next paper.

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